Homework 5

Due Thursday, October 8

⋆ 1. Define an *ultrafilter* of formulas to be a set Γ such that:

- if $\varphi \in \Gamma$ and $\varphi \models \psi$, then $\psi \in \Gamma$,
- if $\varphi \in \Gamma$ and $\psi \in \Gamma$, then $\varphi \wedge \psi \in \Gamma$,
- $\neg\varphi \in \Gamma$ iff $\varphi \notin \Gamma$

Show that a maximally consistent set of sentences is the same thing as an ultrafilter of formulas.

2. Consider a first-order language, with relation symbols $<$ and $=$, and constant symbol 0. Consider the interpretation: the natural numbers, with "less-than" and "equality", and the distinguished element 0. Formalize the following statements:

   (a) "x is less than or equal to y"

   (b) "0 is the smallest number"

   (c) "there is a smallest number"

   (d) "there is no largest number"

   (e) "every number has an immediate successor" (in other words, for every number, there is another one that is the "next largest")

   (f) "every number is greater than some (other) number"

   (g) "there is some number that every (other) number is greater than"

   (h) Add the operations $+$ and $\cdot$, and the constant 1, and formalize the following:

      i. "all square numbers are positive"

      ii. "there's just one even prime number"

      iii. "between every two squares there is always a prime"

3. Consider a first-order language, with predicate symbols M and G, and constant symbol s. The intended interpretation is all people, living or dead, with $M(x)$ meaning "x is mortal", $G(x)$ meaning "x is Greek", and s meaning Socrates. Formalize the following statements:

(a) "if x is Greek, then x is mortal"

(b) "all Greeks are mortal"

(c) "some Greeks are mortal"

(d) "no Greeks are mortal"

(e) "no Greeks are immortal"

(f) "Socrates is a mortal Greek, but there are some Greeks who are immortal"

(g) "if anyone is mortal, Socrates is"

(h) "if all Greeks are mortal, and Socrates is Greek, then Socrates is mortal"

4. Consider a first-order language, with predicate symbols L, P, M and F. The intended interpretation is all people, living or dead, with $L(x, y)$ meaning "x loves y", $P(x, y)$ meaning "x is a parent of y", $M(x)$ meaning "x is male", $F(x)$ meaning "x is female". Formalize the following statements:

   (a) "everyone loves their grandmother"

   (b) "some fathers love women who are not the mothers of their (the fathers') children"

   (c) "not all aunts are loved"

   (d) "to love and to be loved, that is to be a parent!"

5. Fix a language, L, which has two predicate symbols A and B. For each of the following formulas, find an interpretation that makes it true and one that makes it false. Justify your answers.

   (a)
   $$\forall x(A(x) \vee B(x))$$

   (b)
   $$\forall x A(x) \vee \forall x B(x)$$

   (c)
   $$\exists x(A(x) \vee B(x))$$

   (d)
   $$\exists x A(x) \vee \exists x B(x)$$

   and the same for $\wedge$ in place of $\vee$.