Recall the following **Well-Definedness Principle** for quotient sets. Let $X$ be a set with an equivalence relation, written $x \sim y$. The *quotient set* $X/\sim = \{[x] \mid x \in X\}$ is the set of all *equivalence classes* $[x] = \{y \mid x \sim y\}$.

Suppose given a set $A$, and a function $f : X \to A$ that *respects* $\sim$, in the sense that $f(x) = f(y)$ whenever $x \sim y$. Then there is a (uniquely determined) function on the quotient set, $\tilde{f} : X/\sim \to A$ such that for all $x \in X$, $\tilde{f}([x]) = f(x)$.

1. Recall that a (positive) rational number is an equivalence class of pairs of natural numbers $\frac{n}{m} = [(n, m)]$ (where $m \neq 0$), under the equivalence relation $(n, m) \sim (n', m') \iff n \cdot m' = n' \cdot m$.

   (a) Show that the function $\text{inv} : \mathbb{Q} \to \mathbb{Q}$ giving the multiplicative inverse of a rational number, defined as $\text{inv}(\frac{n}{m}) = (\frac{m}{n})$, is well-defined.

   (b) Define the product of a natural number $k$ and a rational number $\frac{n}{m}$ and by:
   $\quad k \cdot \frac{n}{m} = \frac{k \cdot n}{m}$.

   Is this well-defined?

   (c) Show that the attempted definition of a function $\text{add} : \mathbb{Q} \to \mathbb{Q}$ as $\text{add}(\frac{n}{m}) = n + m$ is not well-defined.
2. The *Lindenbaum-Tarski algebra* of propositional logic $LT(\text{Prop})$ is by definition the quotient set

$$LT(\text{Prop}) = (\text{PROP}/\equiv)$$

of the set $PROP$ of propositional formulas, modulo logical equivalence $\varphi \equiv \psi$ (together with a Boolean algebra structure). The elements of $LT(\text{Prop})$ are thus equivalence classes $[\varphi]$, with $[\varphi] = [\psi]$ iff $\varphi \equiv \psi$. The Boolean operations on $LT(\text{Prop})$ are defined by:

- $1 = [\top]$
- $0 = [\bot]$
- $\neg [\varphi] = [\neg \varphi]$
- $[\varphi] \land [\psi] = [\varphi \land \psi]$
- $[\varphi] \lor [\psi] = [\varphi \lor \psi]$

Prove that this is a Boolean algebra. (Hint: you may use anything proved in class, but cite the results used. The point is to show that the operations are all well-defined and the equations hold.)

3. Do problems 1d,e,f on page 37 of van Dalen.

* 4. On page 37, do problem 3d.