Homework 3

Due Thursday, September 21

1. Do problem 1 on page 27 of van Dalen.

2. Using algebraic means, determine conjunctive and disjunctive normal forms for the following formulas:

$$\neg(p \leftrightarrow q), \quad ((p \to q) \to p) \to p$$

   Use these normal forms to determine whether each formula is a tautology.

3. (a) Show that all of the truth functions (on $\{0,1\}$) can be defined in terms of $\{\to, \bot\}$, i.e. that this is a *functionally complete set of connectives.*

   (b) Show that $\{\to, \vee, \wedge\}$ is not a functionally complete set of connectives.

   (c) Conclude that $\{\to, \vee, \wedge, \leftrightarrow, \top\}$ is not a functionally complete set of connectives. (Hint: define the last two in terms of the others.)

4. Let X be a set with an equivalence relation, written $x \sim y$. Suppose given a set A, and a function $f : X \to A$ that respects $\sim$, in the sense that $f(x) = f(y)$ whenever $x \sim y$. Show that there is a function

$$\overline{f} : X/{\sim} \longrightarrow A$$

   such that for all $x \in X$,

$$\overline{f}([x]) = f(x),$$

   where the *quotient set* $X/{\sim} = \{[x] \mid x \in X\}$ is the set of all *equivalence classes* $[x] = \{y \mid x \sim y\}$.

⋆ 5. Use the foregoing to show that one can define the multiplication of a (positive) rational number $\frac{n}{m}$ by a natural number k by the usual formula,

$$k \cdot \frac{n}{m} = \frac{k \cdot n}{m}.$$

   (Recall that a (positive) rational number is an equivalence class of pairs of natural numbers $\frac{n}{m} = [(n,m)]$, under the equivalence relation $(n,m) \sim (n',m') \iff n \cdot m' = n' \cdot m$.)