Homework 3

Due Thursday, September 17

1. (a) Prove the following proposition, which was stated in class:
   Let X be a set with an equivalence relation, written $x \sim y$. Suppose given a set A and a function $f : X \to A$ that respects $\sim$, in the sense that $f(x) = f(y)$ whenever $x \sim y$. Then there is a function
   $$\overline{f} : X/{\sim} \longrightarrow A$$
   such that for all $x \in X$,
   $$\overline{f}([x]) = f(x).$$
   (Here the *quotient set* $X/{\sim} \; = \{[x] \mid x \in X\}$ is the set of all *equivalence classes* $[x] = \{y \mid x \sim y\}$.)

   (b) Use the foregoing to show that one can define the multiplication of a natural number k by a rational number $\frac{n}{m}$ by the usual formula,
   $$k \cdot \frac{n}{m} = \frac{k \cdot n}{m}.$$
   (Recall that a (positive) rational number is an equivalence class of pairs of natural numbers $\frac{n}{m} = [n, m]$, under the equivalence relation $(n, m) \sim (n', m') \iff n \cdot m' = n' \cdot m$.)

2. Do problem 1 on page 27 of van Dalen.

3. (a) Show that $\{\to, \bot\}$ is a complete set of connectives.

   (b) Show that $\{\to, \vee, \wedge\}$ is not a complete set of connectives.

   (c) Conclude that $\{\to, \vee, \wedge, \leftrightarrow, \top\}$ is not a complete set of connectives. (Hint: define the last two in terms of the others.)

4. Determine conjunctive and disjunctive normal forms for the following formulas:
   $$\neg(p \leftrightarrow q), \quad ((p \to q) \to p) \to p$$
   Use these normal forms to determine whether each formula is a tautology.

⋆ 5. Do problem 14 on page 29 of van Dalen.