Homework 11

Due Thursday, November 30

Note: This homework is optional. Only the 10 best homework scores will be counted.

1. For sets A and B, write $A \preceq B$ to mean that there is an injective function $i : A \to B$, and $A \cong B$ to mean that there is a bijective function $f : A \to B$. Recall that A and B are said to have the same cardinal number, $\sharp A = \sharp B$, if and only if $A \cong B$. We can order the cardinals by setting:

$$\sharp A \leq \sharp B \quad \text{iff and only if} \quad A \preceq B.$$

Show that this ordering relation is well-defined, and that it is a pre-ordering.

2. Consider the natural numbers as a monoid $\mathcal{N} = (\mathbb{N}, +^{\mathcal{N}}, 0^{\mathcal{N}})$ in the obvious way. Show that there is a monoid $\mathcal{P} = (P, +^{\mathcal{P}}, 0^{\mathcal{P}})$ satisfying all the same first-order sentences as $\mathcal{N}$, but having a larger cardinality $\sharp|\mathcal{P}|$. Is there a model of all the same sentences having a smaller cardinality?

3. The language of *linear orders with endpoints* has two constant symbols $0, 1$ and a binary relation symbol, written $x \leq y$. The axioms for linear orders with endpoints are:

reflexivity: $\forall x \; (x \leq x)$,

transitivity: $\forall x, y, z \; ((x \leq y \wedge y \leq z) \to x \leq z)$,

antisymmetry: $\forall x, y \; ((x \leq y \wedge y \leq x) \to x = y)$,

linearity: $\forall x, y \; (x \leq y) \vee (y \leq x)$,

endpoints: $\forall x \; (0 \leq x) \wedge (x \leq 1)$.

Consider the following models of the theory of linear orders with endpoints:

$$\mathcal{I} = ([0,1] \subseteq \mathbb{R}, 0, 1, \leq)$$
(the usual ordering of the unit interval in the reals)

$$\mathcal{N} = (\mathbb{N} \cup \{\infty\}, 0, \infty, \leq)$$
(the usual ordering of the natural numbers, but with a new element ∞ added at infinity)

$$\mathcal{Z} = (\mathbb{Z} \cup \{-\infty, +\infty\}, -\infty, +\infty, \leq)$$
(the usual ordering of the integers, but with new elements $-\infty, +\infty$ added at both ends)

(a) Show that these models are distinguishable in first-order logic by producing sentences that are satisfied by each, but not the others.

(b) Can there be a model that satisfies all the same first-order sentences as $\mathcal{N}$ and is uncountable? Justify your answer!

(c) Can there be a model that satisfies all the same first-order sentences as $\mathcal{I}$ and is countable? Justify your answer!

$\star$ (d) (for Grad Students) Give an explicit model that has cardinality less than that of $\mathcal{I}$ but still satisfies the sentence that you found for $\mathcal{I}$ in part (a).