

## HOMEWORK 10

Due December 8: Practice Final, will not be scored

- Let  $L$  be a language with two unary predicates,  $A$  and  $B$ . Consider the biconditional:

$$\exists x(A(x) \wedge B(x)) \leftrightarrow \exists xA(x) \wedge \exists xB(x).$$

- Show that one direction is valid, using only semantic notions. In particular, your answer should make it clear that you know what “valid” means!
  - Show that the other direction is not valid.
- Find a prenex sentence (i.e. one where all the quantifiers occur up front) equivalent to the following:

$$\neg(\exists x\forall yR(x, y) \rightarrow \forall z(\exists yA(y) \vee B(z)))$$

Prove the equivalence algebraically.

- Give natural deduction proofs of the following formulas (using the 4 quantifier rules, and not defining  $\exists\varphi$  as  $\neg\forall\neg\varphi$ !).
  - $\neg\exists x\varphi(x) \rightarrow \forall x\neg\varphi(x)$
  - $\exists x\neg\varphi(x) \rightarrow \neg\forall x\varphi(x)$
  - $(\exists x\varphi \rightarrow \psi) \rightarrow \forall x(\varphi \rightarrow \psi)$ , where  $x$  is not free in  $\psi$ .
- Formalize the following argument in first-order logic, and determine whether it is valid (justify your answer).

Some Greeks are not philosophers.  
 No slaves are philosophers.  
 Therefore, some Greeks are not slaves.

- The following problems concern first-order logic.
  - State the Model Existence Lemma.
  - State the Completeness Theorem.

- (c) Assuming the Model Existence Lemma, prove the Completeness Theorem.
6. The language of *linear orders with endpoints* has two constant symbols  $0, 1$  and a binary relation symbol, written  $x \leq y$ . The axioms for linear orders with endpoints are:

reflexivity:  $\forall x (x \leq x)$ ,

transitivity:  $\forall x, y, z ((x \leq y \wedge y \leq z) \rightarrow x \leq z)$ ,

antisymmetry:  $\forall x, y ((x \leq y \wedge y \leq x) \rightarrow x = y)$ ,

linearity:  $\forall x, y (x \leq y) \vee (y \leq x)$ ,

endpoints:  $\forall x (0 \leq x) \wedge (x \leq 1)$ .

Consider the following models of the theory of linear orders with endpoints:

$\mathcal{I} = ([0, 1] \subseteq \mathbb{R}, 0, 1, \leq)$

(the usual ordering of the unit interval in the reals)

$\mathcal{N} = (\mathbb{N} \cup \{\infty\}, 0, \infty, \leq)$

(the usual ordering of the natural numbers,

but with a new element  $\infty$  added at infinity)

- (a) Show that these models are distinguishable in first-order logic by producing a sentence that is satisfied by one but not the other.
- (b) A theory  $\mathbb{T}$  is said to be *complete* if for every sentence  $\alpha$ , either  $\mathbb{T} \vdash \alpha$  or  $\mathbb{T} \vdash \neg\alpha$  and not both. Is the theory of linear orders with endpoints complete?
- (c) Can there be a model that satisfies all the same first-order sentences as  $\mathcal{N}$  and is uncountable? Justify your answer!
- ★ 7. (for Grad Students)

State and prove the Compactness Theorem for first-order logic. (You may assume other results proved in class.)