1. Let $L$ be a language with two unary predicates, $A$ and $B$. Consider the equivalence
\[ \exists x (A(x) \land B(x)) \leftrightarrow \exists x A(x) \land \exists x B(x). \]
(a) Show that one direction is valid, using direct semantic arguments as in Lemma 3.4.5.
(b) Show explicitly that the other direction is not valid.

2. Show that, in general, $\forall x \varphi(x) \rightarrow \psi \vdash \forall x (\varphi(x) \rightarrow \psi)$ does not hold, even assuming that $\psi$ has no free $x$. Exhibit specific formulas $\varphi(x)$ and $\psi$ for which it fails (and show that this is so).

3. Given any language $L$ and any $L$-structure $A$, consider the “total theory of $A$”,
\[ T(A) = \{ \sigma \mid A \models \sigma \}. \]
Show that $T(A)$ is maximally consistent. Is it always Henkin?

4. Suppose given a language $L = (R, f, c)$ with a two-place relation symbol, one-place function symbol, and a constant symbol, and let $A$ be an $L$-structure. Suppose that there is an equivalence relation $x \sim y$ on $A = |A|$ such that:
\[ a \sim b \text{ and } c \sim d \text{ implies } A \models R(\bar{a}, \bar{c}) \rightarrow R(\bar{b}, \bar{d}) \]
\[ a \sim b \text{ implies } A \models f(\bar{a}) = f(\bar{b}) \]
Define a new $L$-structure $A/\sim$ as follows:
(a) $|A/\sim| = |A|/\sim$, the set of equivalence classes $[a]$ for $a \in A$.
(b) $R^{A/\sim} = \{ [[a], [b]] \mid (a, b) \in R^A \}$
(c) $f^{A/\sim}([a]) = [f^A(a)]$
(d) $c^{A/\sim} = [c^A]$
Show that this is an $L$ structure, i.e. that the specifications are well-defined.

* 5. Referring to the foregoing problem, show that for any sentence $\sigma$,
\[ A \models \sigma \text{ implies } A/\sim \models \sigma \]