HOMEWORK 10

Due Thursday, November 29

1. Determine whether the following syllogism is valid (and justify your answer).

> Some Greeks are not slaves.
> No slaves are women.
> Therefore, some women are not Greek.

2. Using just the language of equality $=$, give formulas expressing the following conditions on structures (i.e. sets) A:

   (a) A is not empty.

   (b) A has at least n elements (for an arbitrary natural number n).

   (c) A has at most n elements (for an arbitrary natural number n).

   (d) Can you find a sentence that expresses that A is infinite?

   (e) Can you find a theory $\mathbb{T}$ (a set of sentences) that expresses this?

3. Consider the natural numbers as a monoid $\mathcal{N} = (\mathbb{N}, +^{\mathcal{N}}, 0^{\mathcal{N}})$ in the obvious way. Show that there is a monoid $\mathcal{P} = (P, +^{\mathcal{P}}, 0^{\mathcal{P}})$ satisfying all the same first-order sentences as $\mathcal{N}$, but having a larger cardinality $\sharp P > \sharp \mathbb{N}$. Is there a model having a smaller cardinality?

4. The language of *linear orders with endpoints* has two constant symbols $0, 1$ and a binary relation symbol, written $x \leq y$. The axioms for linear orders with endpoints are:

   reflexivity: $\forall x \ (x \leq x)$,

   transitivity: $\forall x, y, z \ ((x \leq y \wedge y \leq z) \rightarrow x \leq z)$,

   antisymmetry: $\forall x, y \ ((x \leq y \wedge y \leq x) \rightarrow x = y)$,

   linearity: $\forall x, y \ (x \leq y) \vee (y \leq x)$,

   endpoints: $\forall x \ (0 \leq x) \wedge (x \leq 1)$.

Consider the following models of the theory of linear orders with end-points:

$$\mathcal{I} = ([0,1] \subseteq \mathbb{R}, 0, 1, \leq)$$
(the usual ordering of the unit interval in the reals)

$$\mathcal{N} = (\mathbb{N} \cup \{\infty\}, 0, \infty, \leq)$$
(the usual ordering of the natural numbers,
but with a new element ∞ added at infinity)

$$\mathcal{Z} = (\mathbb{Z} \cup \{-\infty, +\infty\}, -\infty, +\infty, \leq)$$
(the usual ordering of the integers,
but with new elements $-\infty, +\infty$ added at both ends)

(a) Show that these models are distinguishable in first-order logic by producing sentences that are satisfied by each, but not the others.

(b) Can there be a model that satisfies all the same first-order sentences as $\mathcal{N}$ and is uncountable? Justify your answer!

(c) Can there be a model that satisfies all the same first-order sentences as $\mathcal{I}$ and is countable? Justify your answer!

⋆(d) (for Grad Students) Give an explicit model that has cardinality less than that of $\mathcal{I}$ but still satisfies the sentence that you found for $\mathcal{I}$ in part (a).