

HOMEWORK 1
Due Thursday 7 September

Undergraduates should do only the unstarred problems. Graduate students should also do the starred problem.

1. Prove by induction that if $n \geq 4$, then $n! > 2^n$.
(Recall that the *factorial function* is defined by $n! = n \cdot (n - 1) \cdot \dots \cdot 1$.)
2. Show that for inductively defined sets, $C_* \subseteq C^*$ where the former is the set of elements having a construction sequence and the latter is the intersection of all inductive sets.
3. Define the set of “babble-strings” inductively, as follows:
 - the expression “ba” is a babble-string
 - if s is a babble-string, so is “ab” $\hat{\ }s$
 - if s and t are babble-strings, so is $s\hat{\ }t$

Here $\hat{\ }$ is the *concatination operation*: $x\hat{\ }y = xy$.

Prove by induction that every babble-string has the same number of a 's and b 's, and that every babble-string ends with an “a”.

4. Referring to the previous problem, show that the set B of babble-strings is not freely generated. Give a different specification of B such that B is freely generated. Use that specification to define a length function $f : B \rightarrow \mathbb{N}$ giving the number of letters in the string.
- ★ 5. The set of (unlabelled, binary) trees is defined inductively as follows:
 - $*$ is a tree
 - if s, t are trees, so is $[s, t]$

Define functions on trees that count the height of a tree and the width of a tree (the latter is the same as the number of branch-ends). Prove that the height of a tree is always less than or equal to the width of the tree.