Undergraduates should do only the unstarred problems. Graduate students should also do the starred problem.

1. Prove by induction that if \( n \geq 4 \), then \( n! > 2^n \).
(Recall that the factorial function is defined by \( n! = n \cdot (n - 1) \cdot \ldots \cdot 1 \).)

2. Show that for inductively defined sets, \( C \subseteq C^* \) where the former is the set of elements having a construction sequence and the latter is the intersection of all inductive sets.

3. Define the set of “babble-strings” inductively, as follows:
   - “ba” is a babble-string
   - if \( s \) is a babble-string, so is “ab”\( \hat{s} \)
   - if \( s \) and \( t \) are babble-strings, so is \( s \hat{t} \)

   Here \( \hat{} \) is the concatenation operation: \( x \hat{y} = xy \).

   Prove by induction that every babble-string has the same number of \( a \)'s and \( b \)'s, and that every babble-string ends with an “a”.

4. Referring to the previous problem, show that the set \( B \) of babble-strings is not freely generated. Give a different specification of \( B \) such that \( B \) is freely generated. Use that specification to define a length function \( f : B \rightarrow \mathbb{N} \) giving the number of letters in the string.

\* 5. The set of (unlabelled, binary) trees is defined inductively as follows:
   - * is a tree
   - if \( s, t \) are trees, so is \( [s, t] \)

   Define functions on trees that count the height of a tree and the width of a tree (the latter is the same as the number of branch-ends). Prove that the height of a tree is always less than or equal to the width of the tree.