Homework 1

Due Thursday 14 September

Undergraduates are to do only the unstarred problems. Graduate students should also do the starred problem.

1. Prove by induction that:

$$\sum_{i=0}^{n} 2^i \;=\; 2^{n+1} - 1 \;.$$

   (Recall that $\sum_{i=0}^{n} 2^i$ is an abbrevation for $2^0 + 2^1 + 2^2 + \ldots + 2^n$.)

2. Use the least number principle to prove the induction principle.

3. Define the set of "babble-strings" inductively, as follows:

   - "ba" is a babble-string
   - if s is a babble-string, so is "ab"$\hat{}s$
   - if s and t are babble-strings, so is $s\hat{}t$

   Prove by induction that every babble-string has the same number of a's and b's, and that every babble-string ends with an "a".

4. Referring to the previous problem, show that the set B of babble-strings is not freely generated. Give a different specification of B such that B is freely generated. Use that specification to define a length function $f : B \to \mathbb{N}$ giving the number of letters in the string.

5. Write down explicit definitions of the functions f and g, defined recursively by:

   a. $f(0) = 0$, and $f(n+1) = 3 + f(n)$,

   b. $g(0) = 1$, and $g(n+1) = (n+1)^2 g(n)$.
   (Hint: use "factorial" notation: $n! = 1 \cdot 2 \cdot \ldots \cdot n$.)

⋆ 6. Prove by induction that if $n \geq 5$, then $2^n > n^2$.