§ 1.2 INDUCTION AND RECURSION

Induction

There is one special type of construction which occurs frequently both in logic and in other branches of mathematics. We may want to construct a certain subset of a set U by starting with some initial elements of U, and applying certain operations to them over and over again. The set we seek will be the smallest set containing the initial elements and closed under the operations. Its members will be those elements of U which can be built up from the initial elements by applying the operations a finite number of times.

In the special case of immediate interest to us, U is the set of expressions, the initial elements are the sentence symbols, and the operations are $\mathcal{E}_\neg$, $\mathcal{E}_\wedge$, etc. The set to be constructed is the set of wffs. But we will encounter other special cases later, and it will be helpful to view the situation abstractly here.

To simplify our discussion, we will consider an initial set $B \subseteq U$ and a class $\mathcal{F}$ of functions containing just two members f and g, where

$$f : U \times U \to U \quad \text{and} \quad g : U \to U.$$

Thus f is a binary operation on U and g is a unary operation. (Actually $\mathcal{F}$ need not be finite; it will be seen that our simplified discussion here is, in fact, applicable to a more general situation. $\mathcal{F}$ can be any set of relations on U, and in Chapter 2 this greater generality will be utilized. But the case discussed here is easier to visualize and is general enough to illustrate the ideas. For a less restricted version, see Exercise 3.)

If B contains points a and b, then the set C we wish to construct will contain, for example,

$$b, f(b, b), g(a), f(g(a), f(b, b)), g(f(g(a), f(b, b))).$$

Of course these might not all be distinct. The idea is that we are given certain bricks to work with, and certain types of mortar, and we want C to contain just the things we are able to build.

In defining C more formally, we have our choice of two definitions. We can define it "from the top down" as follows: Say that a subset S of U is *closed* under f and g iff whenever elements x and y belong to S, then so do $f(x, y)$ and $g(x)$. Say that S is *inductive* iff $B \subseteq S$ and S is closed under f and g. Let C^* be the intersection of all the inductive subsets of U; thus

$x \in C^*$ iff x belongs to every inductive subset of U. It is not hard to see (and the reader should check) that C^* is itself inductive. Furthermore, C^* is the smallest such set, being included in all the other inductive sets.

The second (and equivalent) definition works "from the bottom up." We want C_* to contain the things which can be reached from B by applying f and g a finite number of times. Temporarily define a *construction sequence* to be a finite sequence $\langle x_0, \ldots, x_n \rangle$ of elements of U such that for each $i \leq n$ we have at least one of

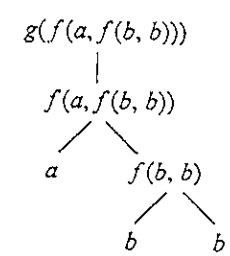
$$x_i \in B,$$
$$x_i = f(x_j, x_k) \qquad \text{for some} \quad j < i,\ k < i,$$
$$x_i = g(x_j) \qquad \text{for some} \quad j < i.$$

Then let C_* be the set of all points x such that some construction sequence ends with x.

Let C_n be the set of points x such that some construction sequence of length n ends with x. Then $C_1 = B$,

$$C_1 \subseteq C_2 \subseteq C_3 \subseteq \cdots,$$

and $C_* = \bigcup_n C_n$. For example, $g(f(a, f(b, b)))$ is in C_5 and hence in C_*, as can be seen by contemplating the tree shown:



We obtain a construction sequence for $g(f(a, f(b, b)))$ by squashing this tree into a linear ordering.

EXAMPLES 1. The natural numbers. Let U be the set of all real numbers, and let $B = \{0\}$. Take one operation S, where $S(x) = x + 1$. Then

$$C_* = \{0, 1, 2, \ldots\}.$$

The set C_* of natural numbers contains exactly those numbers obtainable from 0 by applying the successor operation repeatedly.

Excerpted from Herbert Enderton, A Mathematical Introduction to Logic, Academic Press, 1972

2. The integers. Let U be the set of all real numbers; let $B = \{0\}$. This time take two operations, the successor operation S and the predecessor operation P:

$$S(x) = x + 1 \quad \text{and} \quad P(x) = x - 1.$$

Now C_* contains all the integers,

$$C_* = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

Notice that there is more than one way of obtaining 2 as a member of C_*. For 2 is $S(S(0))$, but it is also $S(P(S(S(0))))$.

3. The algebraic functions. Let U contain all functions whose domain and range are each sets of real numbers. Let B contain the identity function and all constant functions. Let $\mathcal{F}$ contain the operations (on functions) of addition, multiplication, division, and root extraction. Then C_* is the class of algebraic functions.

4. The well-formed formulas. Let U be the set of all expressions and let B be the set of sentence symbols. Let $\mathcal{F}$ contain the five formula-building operations on expressions: $\mathcal{E}_\neg$, $\mathcal{E}_\wedge$, $\mathcal{E}_\vee$, $\mathcal{E}_\to$, and $\mathcal{E}_\leftrightarrow$. Then C_* is the set of all wffs.

At this point we should verify that our two definitions are actually equivalent, i.e., that $C^* = C_*$.

To show that $C^* \subseteq C_*$ we need only check that C_* is inductive, i.e., that $B \subseteq C_*$ and C_* is closed under the functions. Clearly $B = C_1 \subseteq C_*$. If x and y are in C_*, then we can concatenate their construction sequences and append $f(x, y)$ to obtain a construction sequence placing $f(x, y)$ in C_*. Similarly, C_* is closed under g.

Finally, to show that $C_* \subseteq C^*$ we consider a point in C_* and a construction sequence $\langle x_0, \ldots, x_n\rangle$ for it. By ordinary induction on i, we can see that $x_i \in C^*$, $i \le n$. First $x_0 \in B \subseteq C^*$. For the inductive step we use the fact that C^* is closed under the functions. Thus we conclude that

$$\bigcup_n C_n = C_* = C^* = \bigcap\{S : S \text{ is inductive}\}.$$

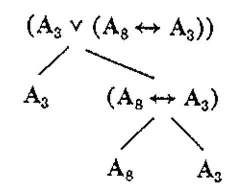
(A parenthetical remark: Suppose our present study is embedded in axiomatic set theory, where the natural numbers are usually defined from the top down. Then our definition of C_* (employing finiteness and hence natural numbers) is not really different from our definition of C^*. But we are not working within axiomatic set theory; we are working within intuitive mathematics. And the notion of natural number seems to be a solid, well-understood intuitive concept.)

Since $C^* = C_*$, we call the set simply C and refer to it as the set *generated from B by* the functions in $\mathcal{F}$. We will often want to prove theorems by using the following:

Induction Principle Assume that C is the set generated from B by the functions in $\mathcal{F}$. If S is a subset of C which includes B and is closed under the functions of $\mathcal{F}$, then $S = C$.

Proof S is inductive, so $C = C^* \subseteq S$. We are given the other inclusion. ∎

The special case now of interest to us is, of course, Example 4. Here C is the class of wffs generated from the set of sentence symbols by the formula-building operations. This special case has interesting features of its own. Both α and β are proper segments of $\mathcal{E}_\wedge(\alpha, \beta)$, i.e., of $(\alpha \wedge \beta)$. More generally, if we look at the family tree of a wff, we see that each constituent is a proper segment of the end product.



Suppose, for example, that we temporarily call an expression *special* if the only sentence symbols in it are among $\{A_2, A_3, A_5\}$ and the only connective symbols in it are among $\{\neg, \to\}$. Then no special wff requires A_9 or $\mathcal{E}_\wedge$ for its construction. In fact, every special wff belongs to the set C_s generated from $\{A_2, A_3, A_5\}$ by $\mathcal{E}_\neg$ and $\mathcal{E}_\to$. (For we can use the induction principle to show that every wff either belongs to C_s or is not special.)

Recursion[1]

We return now to the more abstract case. There is a set U (such as the set of all expressions), a subset B of U (such as the set of sentence symbols),

[1] The reader not already familiar with recursion is advised to postpone reading this subsection until after reading Section 1.3, where a specific application is encountered.

and two functions f and g, where

$$f : U \times U \to U \quad \text{and} \quad g : U \to U.$$

C is the set generated from B by f and g.

The problem we now want to consider is that of defining a function on C recursively. That is, we suppose we are given

1. Rules for computing $\bar{h}(x)$ for $x \in B$.
2a. Rules for computing $\bar{h}(f(x, y))$, making use of $\bar{h}(x)$ and $\bar{h}(y)$.
2b. Rules for computing $\bar{h}(g(x))$, making use of $\bar{h}(x)$.

(For example, this is the situation discussed in Section 1.3, where $\bar{h}$ is the extension of a truth assignment for B.) It is not hard to see that there can be at most one function $\bar{h}$ on C meeting all the given requirements.

But it is possible that no such $\bar{h}$ exists; the rules may be contradictory. For example, let

$$U = \text{the set of real numbers},$$
$$B = \{0\},$$
$$f(x, y) = x \cdot y,$$
$$g(x) = x + 1.$$

Then C is the set of natural numbers. Suppose we impose the following requirements on $\bar{h}$:

1. $\bar{h}(0) = 0$.
2a. $\bar{h}(f(x, y)) = f(\bar{h}(x), \bar{h}(y))$.
2b. $\bar{h}(g(x)) = \bar{h}(x) + 2$.

Then no such function $\bar{h}$ can exist. (Try computing $\bar{h}(1)$, noting that we have both $1 = g(0)$ and $1 = f(g(0), g(0))$.

A similar situation is encountered in algebra.[1] Suppose that you have a group G which is generated from B by the group multiplication and inverse operation. Then an arbitrary map of B into a group H is not necessarily extendible to a homomorphism of the entire group G into H. But if G happens to be a free group with set B of independent generators, then any such map is extendible to a homomorphism of the entire group.

[1] It is hoped that examples such as this will be useful to the reader with some algebraic experience. The other readers will be glad to know that these examples are merely illustrative and not essential to our development of the subject.

Say that C is *freely* generated from B by f and g iff in addition to the requirements for being generated we have

1. f_C and g_C are one-to-one, and
2. The range of f_C, the range of g_C, and the set B are pairwise disjoint.

(Here f_C and g_C are the restrictions of f and g to C.)

Recursion Theorem Assume that the subset C of U is freely generated from B by f and g, where

$$f : U \times U \to U,$$
$$g : U \to U.$$

Further assume that V is a set and F, G, and h functions such that

$$h : B \to V,$$
$$F : V \times V \to V,$$
$$G : V \to V.$$

Then there is a unique function

$$\bar{h} : C \to V$$

such that

(i) For x in B, $\bar{h}(x) = h(x)$.
(ii) For x, y in C,

$$\bar{h}(f(x, y)) = F(\bar{h}(x), \bar{h}(y)),$$
$$\bar{h}(g(x)) = G(\bar{h}(x)).$$

Viewed algebraically, the conclusion of this theorem says that any map h of B into V can be extended to a homomorphism $\bar{h}$ from C (with operations f and g) into V (with operations F and G).

If the content of the recursion theorem is not immediately apparent, try looking at it chromatically. You want to have a function $\bar{h}$ which paints each member of C some color. You have before you

1. h, telling you how to color the initial elements in B;
2. F, which tells you how to combine the color of x and y to obtain the color of $f(x, y)$ (i.e., it gives $\bar{h}(f(x, y))$ in terms of $\bar{h}(x)$ and $\bar{h}(y)$);
3. G, which similarly tells you how to convert the color of x into the color of $g(x)$.

The danger is that of a conflict in which, for example, F is saying "green" but G is saying "red" for the same point (unlucky enough to be equal both to $f(x, y)$ and $g(z)$ for some x, y, z). But if C is freely generated, then this danger is avoided.

EXAMPLES Consider again the examples of the preceding subsection.

1. $B = \{0\}$ with one operation, the successor operation S. Then C is the set N of natural numbers. Since the successor operation is one-to-one, C is freely generated from $\{0\}$ by S. Therefore, by the recursion theorem, for any set V, any $a \in V$, and any $F : V \to V$ there is a unique $\bar{h} : N \to V$ such that $\bar{h}(0) = a$ and $\bar{h}(S(x)) = F(\bar{h}(x))$ for each $x \in N$. For example, there is a unique $\bar{h} : N \to N$ such that $\bar{h}(0) = 0$ and $\bar{h}(S(x)) = 1 - \bar{h}(x)$. This function has the value 0 at even numbers and the value 1 at odd numbers.

2. The integers are generated from $\{0\}$ by the successor and predecessor operations but not freely generated.

3. Freeness fails also for the generation of the algebraic functions in the manner described.

4. The wffs are freely generated from the sentence symbols by the five formula-building operations. The purpose of Section 1.4 is to prove this fact. It follows, for example, that there is a unique function $\bar{h}$ defined on the set of wffs such that

$$\bar{h}(A) = 1 \text{ for a sentence symbol } A,$$
$$\bar{h}((\neg\alpha)) = 3 + \bar{h}(\alpha),$$
$$\bar{h}((\alpha \wedge \beta)) = 3 + \bar{h}(\alpha) + \bar{h}(\beta),$$

and similarly for $\vee$, $\to$, and $\leftrightarrow$. This function gives the length of each wff.

Proof of the recursion theorem The idea is to let $\bar{h}$ be the union of many approximating functions. Temporarily call a function v (which maps part of C into V) *acceptable* if it meets the conditions imposed on $\bar{h}$ by (i) and (ii). More precisely, v is acceptable iff the domain of v is a subset of C, the range a subset of V, and for any x and y in C:

(i′) If x belongs to B and to the domain of v, then $v(x) = h(x)$.

(ii′) If $f(x, y)$ belongs to the domain of v, then so do x and y, and $v(f(x, y)) = F(v(x), v(y))$. If $g(x)$ belongs to the domain of v, then so does x, and $v(g(x)) = G(v(x))$.

Let K be the collection of all acceptable functions, and let $\bar{h}$ be the union of K. Thus

$$\langle x, y\rangle \in \bar{h} \qquad \text{iff } v(x) = y \text{ for some } v \in K.$$

We claim that $\bar{h}$ meets our requirements. We will outline the procedure for checking this, leaving many details to the reader. (We feel that a detailed understanding of this set-theoretic proof, while nice, is not essential here. But some understanding of its outline should make the theorem itself more comprehensible.)

1. We claim that $\bar{h}$ is a function. Let

$$S = \{x \in C\colon \text{For at most one } y, \langle x, y\rangle \in \bar{h}\}.$$

It is easy to verify that S is inductive, by using (i′) and (ii′). Hence $S = C$ and $\bar{h}$ is a function.

2. We claim that $\bar{h} \in K$; i.e., that $\bar{h}$ is an acceptable function. This follows fairly easily from the definition of $\bar{h}$ and the fact that it is a function.

3. We claim that $\bar{h}$ is defined throughout C. It suffices to show that the domain of $\bar{h}$ is inductive. It is here that the assumption of freeness is used. For example, one case is the following: Suppose that x is in the domain of $\bar{h}$. Then $\bar{h}$; $\langle g(x), G(\bar{h}(x))\rangle$ is acceptable. (The freeness is required in showing that it is acceptable.) Consequently, $g(x)$ is in the domain of $\bar{h}$.

4. We claim that $\bar{h}$ is unique. For given two such functions, let S be the set on which they agree. Then S is inductive, and so equals C. ■

It is interesting to note that there is an alternative way of describing the proof of the recursion theorem, by presenting the desired function $\bar{h}$ as the set (of pairs) generated from a set by some operations. For let

$$\hat{U} = U \times V,$$
$$\hat{B} = h, \qquad \text{a subset of } \hat{U},$$
$$\hat{F}(\langle x, u\rangle, \langle y, v\rangle) = \langle f(x, y),\ F(u, v)\rangle,$$
$$\hat{G}(\langle x, u\rangle) = \langle g(x),\ G(u)\rangle.$$

Thus $\hat{F}$ is the binary operation on $\hat{U}$ obtained as the product of the operations f and F. Now let $\bar{h}$ be the subset of $\hat{U}$ generated from $\hat{B}$ by $\hat{F}$ and $\hat{G}$. Then it can be checked that $\bar{h}$ has the desired properties. The freeness must be used in showing that $\bar{h}$ is a function.

One final comment on induction and recursion: The induction principle we have stated is not the only one possible. It is entirely possible to give proofs by induction (and definitions by recursion) on the length of expressions, the number of places at which connective symbols occur, etc. Such methods are inherently less basic but may be necessary in some situations.

EXERCISES

1. Suppose that C is generated from a set $B = \{a, b\}$ by the binary operation f and unary operation g. List all the members of C_2. How many members might C_3 have? C_4?

2. Obviously $(\mathbf{A}_3 \rightarrow \wedge \mathbf{A}_4)$ is not a wff. But prove that it is not a wff.

3. We can generalize the discussion in this section by requiring of $\mathscr{F}$ only that it be a class of relations on U. C_* is defined as before, except that $\langle x_0, x_1, \ldots, x_n \rangle$ is now a construction sequence provided that for each $i \leq n$ we have either $x_i \in B$ or $\langle x_{j_1}, \ldots, x_{j_k}, x_i \rangle \in R$ for some $R \in \mathscr{F}$ and some $j_1, \ldots, j_k$ all less than i. Give the correct definition of C^* and show that $C^* = C_*$.

§ 1.3 TRUTH ASSIGNMENTS

We want to define what it means for one wff of our language to follow logically from other wffs. For example, $\mathbf{A}_1$ should follow from $(\mathbf{A}_1 \wedge \mathbf{A}_2)$. For no matter how the parameters $\mathbf{A}_1$ and $\mathbf{A}_2$ are translated back into English, if the translation of $(\mathbf{A}_1 \wedge \mathbf{A}_2)$ is true, then the translation of $\mathbf{A}_1$ must be true. But the notion of all possible translations back into English is unworkably vague. Luckily the spirit of this notion can be expressed in a simple and precise way.

Fix once and for all a set $\{T, F\}$ of *truth values* consisting of two distinct points:

T, called *truth*,
F, called *falsity*.

(It makes no difference what these points themselves are; they might as well be the numbers 1 and 0.) Then a *truth assignment* v for a set $\mathscr{S}$ of sentence symbols is a function

$$v : \mathscr{S} \rightarrow \{T, F\}$$

assigning either T or F to each symbol in $\mathscr{S}$. These truth assignments will be used in place of the translations into English mentioned in the preceding paragraph.

(At this point we have committed ourselves to *two-valued* logic. It is also possible to study three-valued logic, in which case one has a set of three possible truth values. And then, of course, it is a small additional step to allow 512 or $\aleph_0$ truth values; or to take as the set of truth values the unit interval $[0, 1]$ or some other convenient space. A particularly interesting case is that for which the truth values form a complete Boolean algebra. But it is two-valued logic that has always had the greatest significance, and we will be content to confine ourselves to this case.)

Let $\bar{\mathscr{S}}$ be the set of wffs generated from $\mathscr{S}$ by the five formula-building operations. ($\bar{\mathscr{S}}$ can also be characterized as the set of wffs whose sentence symbols are all in $\mathscr{S}$; see the remarks at the end of the subsection on induction in Section 1.2.) We want an extension $\bar{v}$ of v,

$$\bar{v} : \bar{\mathscr{S}} \rightarrow \{T, F\},$$

which assigns the correct truth value to each wff in $\bar{\mathscr{S}}$. It should meet the following conditions:

0. For any $A \in \mathscr{S}$, $\bar{v}(A) = v(A)$. (Thus $\bar{v}$ is an extension of v.)

For any α, β in $\bar{\mathscr{S}}$:

1. $\bar{v}((\neg\alpha)) = \begin{cases} T \text{ if } \bar{v}(\alpha) = F, \\ F \text{ otherwise.} \end{cases}$

2. $\bar{v}((\alpha \wedge \beta)) = \begin{cases} T \text{ if } \bar{v}(\alpha) = T \text{ and } \bar{v}(\beta) = T, \\ F \text{ otherwise.} \end{cases}$

3. $\bar{v}((\alpha \vee \beta)) = \begin{cases} T \text{ if } \bar{v}(\alpha) = T \text{ or } \bar{v}(\beta) = T \text{ (or both)}, \\ F \text{ otherwise.} \end{cases}$

4. $\bar{v}((\alpha \rightarrow \beta)) = \begin{cases} F \text{ if } \bar{v}(\alpha) = T \text{ and } \bar{v}(\beta) = F, \\ T \text{ otherwise.} \end{cases}$

5. $\bar{v}((\alpha \leftrightarrow \beta)) = \begin{cases} T \text{ if } \bar{v}(\alpha) = \bar{v}(\beta), \\ F \text{ otherwise.} \end{cases}$

(Conditions 1–5 are given in tabular form in Table III. It is at this point that the intended meaning of, for example, the conjunction symbol enters into our formal proceedings. Note especially the intended meaning of $\rightarrow$. Whenever α is assigned F, then $(\alpha \rightarrow \beta)$ is considered "vacuously true" and is assigned the value T. For this and the other connectives, it is certainly possible to question how accurately we have reflected the common meaning in everyday speech of "if . . . , then," "or," etc. But our ultimate concern lies more with mathematical statements than with the subtle nuances of everyday speech.)