1.2 Function and Recursion

We obtain a construction sequence for $(p, q, r)$ by applying this: 

$$
\begin{array}{c}
(p,q) \\
(q,r) \\
(p,q) \\
(q,r) \\
(p,q) \\
(q,r)
\end{array}
$$

We can be seen by comparing the tree shown: 

$$
\begin{array}{c}
(p,q) \\
(q,r) \\
(p,q) \\
(q,r) \\
(p,q) \\
(q,r)
\end{array}
$$

and $C = \{ x \}$.

If $C$ contains points $a$ and $b$, then the set $C$ of points $x$ such that some construction sequence of $\langle t \rangle$ ends with $x$ and $\langle t \rangle \leq C$ is $B$. Let $C$ be the set of all points $x$ such that some construction sequence $f$ of $\langle t \rangle$ ends with $x$.

Then let $C_x = \{ x \}$ for some $f$.

$$
\begin{array}{c}
(x) \alpha \\
(x) \beta
\end{array}
$$

For some $x > f$, $x \in C$. For some $x \in C$, for some $x > f$.

We have at least one $x$ such that for each $x \in C$ and any number of times, $f$ can be applied to $x$ and the result of each application of $f$ is not less than $x$.

There is one special type of construction which occurs frequently both
I.9 Induction and Recursion

The natural numbers are the set of all non-negative integers, denoted \( \mathbb{N} \), and include 0. The set of all positive integers, denoted \( \mathbb{N}^+ \), is the set of all non-negative integers excluding 0.

We now have to prove that our previous study is correct.

The principle of mathematical induction is a way to prove that a statement holds for all natural numbers. It consists of two steps:

1. Base Case: Prove that the statement holds for the smallest value (usually 0 or 1).
2. Inductive Step: Assume the statement holds for some arbitrary natural number \( k \) and prove that it also holds for \( k + 1 \).

We start with the principle of mathematical induction to prove that a certain property holds for all natural numbers.

**Proposition:** Suppose our present study is embedded in a set of all expressions of \( L \) such as the set of sentence symbols.

We return now to the more abstract case. There is a \( L \) such as the set of all expression symbols.

**Definition:**

A binary relation \( \subseteq \) on the set \( L \) is called a preorder if it is reflexive and transitive.

**Theorem:** Suppose \( \subseteq \) is a preorder. Then \( \subseteq \) is an equivalence relation if and only if it is a preorder.

**Proof:** Let \( \subseteq \) be a preorder. We want to prove that it is also symmetric.

**Definition:** A relation \( \equiv \) on the set \( L \) is called an equivalence relation if it is reflexive, symmetric, and transitive.

**Theorem:** Suppose \( \equiv \) is an equivalence relation on the set \( L \). Then \( \equiv \) is a preorder.

**Proof:** Let \( \equiv \) be an equivalence relation.

**Example:** The natural numbers are the set of all non-negative integers.

**Theorem:** The set of all natural numbers is closed under the operation of addition.

**Proof:** Let \( a, b \in \mathbb{N} \). Then \( a + b \in \mathbb{N} \).

**Corollary:** The set of all natural numbers is closed under the operation of multiplication.

**Definition:** A function is a special kind of relation that associates each element of a set with exactly one element of another set.

**Example:** Consider the function \( f: \mathbb{N} \to \mathbb{N} \) defined by \( f(n) = n^2 \).

**Theorem:** The function \( f(x) = x^2 \) is not one-to-one.

**Proof:** Let \( x, y \in \mathbb{N} \) and \( x \neq y \). Then \( f(x) = f(y) \).

**Definition:** A function is said to be one-to-one if for every \( x \) in the domain, \( f(x) \) is unique and \( f(x) \neq f(y) \) whenever \( x \neq y \).

**Example:** Consider the function \( g(x) = 2x \).

**Theorem:** The function \( g(x) = 2x \) is one-to-one.

**Proof:** Let \( x, y \in \mathbb{N} \) and \( x \neq y \). Then \( g(x) = g(y) \).

**Definition:** A function is said to be onto if for every \( y \) in the codomain, there exists at least one \( x \) in the domain such that \( f(x) = y \).

**Example:** Consider the function \( h(x) = x^2 \).

**Theorem:** The function \( h(x) = x^2 \) is not onto.

**Proof:** Let \( y \in \mathbb{N} \) and \( y \neq 0 \). Then \( h(x) = y \) for some \( x \in \mathbb{N} \).

**Definition:** A function is said to be bijective if it is both one-to-one and onto.

**Example:** Consider the function \( k(x) = x + 1 \).

**Theorem:** The function \( k(x) = x + 1 \) is bijective.

**Proof:** Let \( x, y \in \mathbb{N} \) and \( x \neq y \). Then \( k(x) = k(y) \).

**Definition:** A function is said to be a bijection if it is one-to-one and onto.

**Example:** Consider the function \( l(x) = x^3 \).

**Theorem:** The function \( l(x) = x^3 \) is a bijection.

**Proof:** Let \( x, y \in \mathbb{N} \) and \( x \neq y \). Then \( l(x) = l(y) \).

**Definition:** A function is said to be a permutation if it is a bijection.

**Example:** Consider the function \( m(x) = x! \).

**Theorem:** The function \( m(x) = x! \) is a permutation.

**Proof:** Let \( x, y \in \mathbb{N} \) and \( x \neq y \). Then \( m(x) = m(y) \).

**Definition:** A function is said to be a recursive function if it is defined in terms of itself.

**Example:** Consider the function \( n(x) = x + 2 \).

**Theorem:** The function \( n(x) = x + 2 \) is recursive.

**Proof:** Let \( x, y \in \mathbb{N} \) and \( x \neq y \). Then \( n(x) = n(y) \).

**Definition:** A function is said to be a computable function if it is defined by a computer program.

**Example:** Consider the function \( o(x) = \text{if } x \text{ is even then } x/2 \text{ else } 3x + 1 \).

**Theorem:** The function \( o(x) = \text{if } x \text{ is even then } x/2 \text{ else } 3x + 1 \) is computable.

**Proof:** Let \( x, y \in \mathbb{N} \) and \( x \neq y \). Then \( o(x) = o(y) \).

**Definition:** A function is said to be a primitive recursive function if it is defined by composition and primitive recursion.

**Example:** Consider the function \( p(x) = x \).

**Theorem:** The function \( p(x) = x \) is primitive recursive.

**Proof:** Let \( x, y \in \mathbb{N} \) and \( x \neq y \). Then \( p(x) = p(y) \).

**Definition:** A function is said to be a general recursive function if it is defined by primitive recursion and minimization.

**Example:** Consider the function \( q(x) = \text{the smallest } y \text{ such that } y > x \).

**Theorem:** The function \( q(x) = \text{the smallest } y \text{ such that } y > x \) is general recursive.

**Proof:** Let \( x, y \in \mathbb{N} \) and \( x \neq y \). Then \( q(x) = q(y) \).
Then there is a unique function

\[ A \rightarrow \mathcal{O} : \psi \]

such that

\[ ((\psi \circ \theta))(x) = (\psi \circ \theta)(x) \]

(i) For all \( x \in \mathcal{O} \),

\[ (\psi \circ \theta)(x) = (\psi \circ \theta)(x) \]

where \( \mathcal{O} \) is a set and \( \psi \) is a function such that

\[ \mathcal{O} \rightarrow \mathcal{O} : \psi \]

This is an example of a function.

Further assume that \( \mathcal{O} \) is a set and \( \mathcal{F} \) is a function such that

\[ \mathcal{O} \rightarrow \mathcal{F} : \theta \]

and \( \theta \) is a function such that

\[ \mathcal{O} \rightarrow \mathcal{O} \times \mathcal{O} : \eta \]

We say that \( \theta \) is freely generated from \( \mathcal{F} \) by \( \eta \) if \( \mathcal{F} \) and \( \eta \) are in addition to the function \( \theta \).

1.2. Induction and Recursion
we have stated, is not the only one possible. It is entirely possible to give
an alternative definition of induction which is based on the fact that the
domain of the comprehension function is the set of its values.

On the other hand, we shall not discuss this point here. The
proof of the induction principle is essentially the same as the
one given above.

Thus if the binary operation on \( \mathcal{G} \) is defined as the
product of the operation on \( \mathcal{G} \) and the
function \( x \mapsto \text{domain of } y \), we have

\[
\text{domain of } y \mapsto \text{domain of } y
\]

Notice that this gives the same result as the previous definition.

\[
\phi(x, y) = \text{domain of } y \mapsto \text{domain of } y
\]

1.2 Induction and Recursion

The set \( \{ x \in \mathbb{N} : x \leq n \} \) is defined in terms of the
successor function and the number 1. However, it can also be defined
in terms of the operation \( \rightarrow \) and the number 0. Thus,

\[
\forall x \in \mathbb{N} \ (x \rightarrow 0 = 1) \land (x \rightarrow (y + 1) = (x \rightarrow y + 1))
\]

1.3 Sentential Logic

The set \( \{ x \in \mathbb{N} : x \leq n \} \) is defined in terms of the
successor function and the number 1. However, it can also be defined
in terms of the operation \( \rightarrow \) and the number 0. Thus,

\[
\forall x \in \mathbb{N} \ (x \rightarrow 0 = 1) \land (x \rightarrow (y + 1) = (x \rightarrow y + 1))
\]
everyday speech)

the more with the mathematical statements than with the subject names of

2.2.3.1. are then "in" or "on" in the phrase concerned line "of" "and"

is written in the section. "Then" is considered "a" composition if ever the two symbols are in the same line. "Then" is considered "a" composition if ever the two symbols are in the same line.

If a symbol is a function

then a symbol is a function of a set of (or, in our example)

which makes no difference when these points expressions are: they might as

be used in place of the two mentioned in the preceding paragraphs

for which the expression in (or, in our example

1.2. TRUTH ASSIGNMENTS

show that $C = C$.

and some $F$, $G$, $F$, $G$. Not only the correct definition of $C$ and

we have either $x \in \mathcal{E}$ or $x \notin \mathcal{E}$, if some $E$, $F$, $G$.

2. Obviously $A \Leftarrow B$ is not a well. But prove that it is not a well.

EXERCISES

with the exception of only a few others, the correctness of $C$. How many

that $C$ is generated from a set $\{f, g\}$ by the binary

products by induction (and definition by recursion) on the length of expressions...