Solutions to Homework #13

4. Suppose $Mod(T_1 \cup T_2) = \emptyset$. By compactness, there is a finite subset of $T_1 \cup T_2$ that has no model. In other words, there is a set $\{\varphi_1, \ldots, \varphi_k\}$ of sentences in T_1, and another set $\{\psi_1, \ldots, \psi_l\}$ of sentences in T_2, such that $\{\varphi_1, \ldots, \varphi_k, \psi_1, \ldots, \psi_l\}$ has no model.

   Let $\sigma = \varphi_1 \wedge \ldots \wedge \varphi_k$. Since each φ_i is in T_1, it is easy to see that $T_1 \models \sigma$. But any structure that satisfies T_2 has to satisfy $\{\psi_1, \ldots, \psi_l\}$, so it can't satisfy all of the φ_i's. So $T_2 \models \neg\sigma$.

5. Suppose $T_1 \neq T_2$. Then there is a sentence that is in one but not the other. Without loss of generality, suppose φ is a sentence in T_1 but not T_2. Since T_2 is a theory, $T_2 \not\models \varphi$, and so $T_2 \cup \{\neg\varphi\}$ is consistent.

   Let $\mathcal{A}$ be a model of $T_2 \cup \{\neg\varphi\}$. Since $\mathcal{A} \models \neg\varphi$, $\mathcal{A}$ is a model of T_2 but not T_1.

8. a. It is easy to verify that the map $f(x) = x + 1$ is an isomorphism.
   b. By Lemma 3.3.3, any two isomorphic structures are elementarily equivalent.
   c. Clearly $|\mathcal{A}| \subseteq |\mathcal{B}|$, and the ordering is the same on both universes.
   d. $\mathcal{A} \models \exists x\ (x < \bar{1})$, but not $\mathcal{B}$.

10. It is easy to verify that $f(x) = 2x$ is an automorphism of this structure. But $1 \times 1 = 1$, while

$$f(1) \times f(1) = 2 \times 2 = 4 \neq f(1).$$

13. Here is the algorithm: on input φ, in parallel look for a proof of φ and a proof of $\neg\varphi$ from the axioms of T. Since T is complete, you are bound to find one or the other. If it is a proof of φ, answer "yes," $\varphi \in T$; otherwise, answer "no," $\varphi \notin T$.

15. I cancelled this question because we did not cover second-order logic in time. But just FYI, note that being well ordered is naturally a second-order property, since we can use a second-order variable to range over subsets:

$$\forall S(\exists x\ S(x) \rightarrow \exists x\ (S(x) \wedge \forall y\ (y < x \rightarrow \neg S(y)))).$$