# Solutions to Homework #10

6. Each line below combines a number of steps. In the first step I rename all variables that are "different," and use the fact that for arbitrary formulas $\eta$ and $\theta$, $\neg(\eta \rightarrow \theta)$ is equivalent to $\eta \wedge \neg\theta$. In the last step, there is flexibility in the order in which you bring quantifiers to the front.

$$\neg(\exists x\ \varphi(x,y) \wedge (\forall y\ \psi(y) \rightarrow \varphi(x,x)) \rightarrow \exists x\ \forall y\ \sigma(x,y))$$

$$\begin{aligned}
&\equiv\ \exists z\ \varphi(z,y) \wedge (\forall w\ \psi(w) \rightarrow \varphi(x,x)) \wedge \neg\exists u\ \forall v\ \sigma(u,v) \\
&\equiv\ \exists z\ \varphi(z,y) \wedge (\exists w\ \neg\psi(w) \vee \varphi(x,x)) \wedge \forall u\ \exists v\ \neg\sigma(u,v) \\
&\equiv\ \exists z\ \varphi(z,y) \wedge \exists w\ (\neg\psi(w) \vee \varphi(x,x)) \wedge \forall u\ \exists v\ \neg\sigma(u,v) \\
&\equiv\ \exists z, w\ \forall u\ \exists v\ (\varphi(z,y) \wedge (\neg\psi(w) \vee \varphi(x,x)) \wedge \neg\sigma(u,v)).
\end{aligned}$$

7. Let $\mathcal{M}$ be any structure. I need to show that

$$\mathcal{M} \models \exists x\ (\varphi(x) \rightarrow \forall y\ \varphi(y)).$$

Applying Lemma 2.4.5, this amounts to showing that there is an element $a$ in the universe of $\mathcal{M}$ such that either $\mathcal{M} \models \neg\varphi(\bar{a})$ or $\mathcal{M} \models \forall y\ \varphi(y)$.

If $\mathcal{M} \models \forall y\ \varphi(y)$, then we are done; any element $a$ of the universe of $\mathcal{M}$ will do. (Remember, we are assuming that our structures have nonempty universes.) Otherwise, $\mathcal{M} \not\models \forall y\ \varphi(y)$, which is to say, it is not the case that for every $b$ in $|\mathcal{M}|$, $\mathcal{M} \models \varphi(\bar{b})$. In other words, for some $b$ in $|\mathcal{M}|$, $M \models \neg\varphi(\bar{b})$. In that case, we can take $a = b$, and again we're done.

Alternatively, you can use the transformations in Section 2.5 to show that the following are all equivalent:

$$\begin{aligned}
&\exists x\ (\varphi(x) \rightarrow \forall y\ \varphi(y)) \\
&\exists x\ (\neg\varphi(x) \vee \forall y\ \varphi(y)) \\
&\exists x\ \neg\varphi(x) \vee \forall y\ \varphi(y) \\
&\neg\forall x\ \varphi(x) \vee \forall y\ \varphi(y) \\
&\neg\forall x\ \varphi(x) \vee \forall x\ \varphi(x)
\end{aligned}$$

Clearly the last formula is valid.

11. a. $\forall x\ (Cow(x) \rightarrow EatsGrass(x))$
    b. $\exists x\ (Car(x) \wedge Blue(x) \wedge Old(x))$
    c. $\neg\exists x\ (Car(x) \wedge \neg Pink(x))$
    d. $\forall x\ (Car(x) \wedge Old(x) \rightarrow MustBeInspectedAnnually(x))$

14. To separate $\mathfrak{A}_1$ and $\mathfrak{A}_2$, let $\sigma$ say that there is a smallest element:

$$\exists x\ \forall y\ (x \leq y).$$

To separate $\mathfrak{A}_2$ and $\mathfrak{B}$, let $\sigma$ say that between any two elements, there is another element:

$$\forall x, y\ (x < y \rightarrow \exists z\ (x < z \wedge z < y))$$

where $x < y$ abbreviates $x \leq y \wedge \neg(x = y)$.

15. $\sigma$ "says" that there is something that is comparable with every element. Let $\mathfrak{A}$ and $\mathfrak{B}$ be the posets below:

In other words, $\mathfrak{B}$ is the poset with two incomparable elements, and $\mathfrak{A}$ is the poset with two elements, one of which is less than the other. Then $\mathfrak{A} \models \sigma$ and $\mathfrak{B} \models \neg\sigma$. (For $\mathfrak{A}$, the 1-element poset would also work.)

19. a. $Prime(x) \equiv x \neq 0 \wedge x \neq S(0) \wedge \forall y, z\ (x = y \times z \rightarrow y = x \vee z = x)$
    b. $OddSquare(x) \equiv \exists y\ (x = y \times y) \wedge \exists z\ (x = (z + z) + 1)$
    c. $ThreePrimeFactors(x) \equiv \exists u, v, y, z\ (x = u \times v \times y \times z \wedge Prime(u) \wedge Prime(v) \wedge Prime(y) \wedge u \neq v \wedge u \neq y \wedge v \neq y)$

    Once again, there are many reasonable variations of these.