Corollary 7.2.4 *Each of the class of finite groups, finite rings, finite partial orders, etc. is not definable in first-order logic.*

A few simple observations will allow us to answer another question from Chapter 5. Note that if a class of structures is definable by a finite set of sentences $\{\varphi_1, \ldots, \varphi_k\}$, then it is defined by a single sentence $\varphi_1 \wedge \ldots \wedge \varphi_k$. And if a class of structures is defined by a single sentence σ, the complement of that class is defined by $\neg\sigma$. So we can conclude, for example,

Corollary 7.2.5 *The class of infinite structures in the language of equality cannot be defined by any finite set of sentences.*

If it could, then the class of finite structures would be definable, contrary to the corollary above. (You can find additional information on "finite axiomatizability" on page 116 of van Dalen.)

For another application of the compactness theorem, let us consider an example from graph theory. Remember that a graph is a structure $\mathcal{A} = \langle A, R\rangle$ satisfying

$$\forall x\ \neg R(x,x) \wedge \forall x, y\ (R(x,y) \rightarrow R(y,x)).$$

(As usual, I am being bad by using R for both the symbol in the first-order language and for the relation it denotes in $\mathcal{A}$. I trust that by now you can tell the difference.) The elements of A are called the *vertices* of the graph, and, if a and b are vertices, we say that there is an *edge* between a and b if and only if $R(a,b)$ holds. A *path* from a to b is a (finite!) sequence of vertices $a_1, \ldots, a_n$ such that $a_1 = a$, $a_n = b$, and for each $i < n$, $R(a_i, a_{i+1})$. A graph is said to be *connected* if there is a path between every two vertices.

Theorem 7.2.6 *The class of connected graphs is not definable in first-order logic.*

Proof. In other words, the theorem says that there is no set of sentences Γ in the language of graph theory such that the models of Γ are exactly the connected graphs. For the sake of contradiction, let us suppose otherwise; i.e. suppose Γ *does* define the class of connected graphs.

First, note that with first-order logic, it *is* possible to write down a formula $\varphi_n(x,y)$ which says "there is a path of length n from x to y". The following does the trick:

$$\exists z_1, \ldots, z_n\ (z_1 = x \wedge z_n = y \wedge R(z_1, z_2) \wedge \ldots \wedge R(z_{n-1}, z_n)).$$

Pick two new constants, c and d, and add them to the language. Let Δ be the following set of sentences:

$$\Gamma \cup \{\neg\varphi_0(c,d), \neg\varphi_1(c,d), \ldots\}$$

I claim that every finite subset of Δ is satisfiable. To see this, let Δ' be a finite subset of Δ. Then for some n, Δ' is included in the set

$$\Gamma \cup \{\neg\varphi_0(c,d), \ldots, \neg\varphi_n(c,d)\}.$$

To get a model of Δ', one only need find a connected graph with two elements that are not connected by a path of length $n+1$, and then let c and d denote these two elements. For example, one can just cook up a graph $a_1, \ldots, a_{n+1}$ with an edge from each a_i to a_{i+1} and nothing more, let c denote a_1, and let d denote a_{n+1}.

By compactness, Δ has a model. Let $\mathcal{A}$ be the reduct of this model to the original language (just drop the denotions of c and d). Since $\mathcal{A}$ satisfies Γ, it is a connected graph. On the other hand, since the expanded structure models

$$\{\neg\varphi_0(c,d), \ldots, \neg\varphi_n(c,d)\}$$

there is no path between the elements denoted by c and d, a contradiction. $\square$

A variation of this trick allows us to show that things like the class of torsion groups and well-orderings are not definable. If you don't know what these are, don't worry about it. Let us now use a similar argument to "construct" a nonstandard model of arithmetic.

Theorem 7.2.7 *Let $\mathfrak{N}$ be the structure $\langle \mathbb{N}, 0, S, +, \times, < \rangle$. There is a structure $\mathfrak{M}$ such that*

- *$\mathfrak{M}$ is elementarily equivalent to $\mathfrak{N}$*
- *$\mathfrak{M}$ is not isomorphic to $\mathfrak{N}$*

Proof. Let L be the language of $\mathfrak{N}$, and L' be L together with a new constant c. Let Γ be the following set of sentences in L':

$$Th(\mathfrak{N}) \cup \{0 < c, S(0) < c, S(S(0)) < c, \ldots\}.$$

Every finite subset Γ' of Γ has a model of the form $\langle \mathbb{N}, 0, S, +, \times, <, m \rangle$, where m is a natural number that is large enough to satisfy the finitely many sentences involving c in Γ'. By compactness, Γ has a model,

$$\mathfrak{A} = \langle A, 0^{\mathfrak{A}}, S^{\mathfrak{A}}, +^{\mathfrak{A}}, \times^{\mathfrak{A}}, <^{\mathfrak{A}}, c^{\mathfrak{A}} \rangle.$$