Homework #6

Due Wednesday, October 3

1. Read Section 1.4 in van Dalen, and start reading Section 1.5.

⋆ 2. Use our semantic definitions to prove or find a counterexample to each of the following:

   a. For every set of formulas Γ, every formula φ, and every formula ψ, if $\Gamma \models \varphi \wedge \psi$, then $\Gamma \models \varphi$ and $\Gamma \models \psi$.

   b. For every set of formulas Γ, every formula φ, and every formula ψ, if $\Gamma \models \varphi \vee \psi$, then $\Gamma \models \varphi$ or $\Gamma \models \psi$.

⋆ 3. Do problems 4 and 5 on page 28 of van Dalen. In other words, if $\varphi \mid \psi$, read "φ nand ψ," means that φ and ψ are not both true, and $\varphi \downarrow \psi$, read "φ nor ψ," means that neither φ nor ψ is true, show that $\{|\}$ and $\{\downarrow\}$ are complete sets of connectives.

4. Do problem 6 on page 28. In other words, show that these are the only two binary connectives that have this property.

5. Show that $\{\rightarrow, \bot\}$ is a complete set of connectives.

⋆ 6.

   a. Show that $\{\rightarrow, \vee, \wedge\}$ is not a complete set of connectives. (Hint: show that any formula involving only these connectives is true when all the variables are true.)

   b. Conclude that $\{\rightarrow, \vee, \wedge, \leftrightarrow, \top\}$ is not a complete set of connectives. (Hint: define the last two in terms of the others.)

7. a. Show that $\{\bot, \leftrightarrow\}$ is not a complete set of connectives. (Hint: show that any formula involving only these connectives and the variables p_0 and p_1 is equivalent to one of the following: $\bot$, $\top$, p_0, p_1, $\neg p_0$, $\neg p_1$, $p_0 \leftrightarrow p_1$, or $p_0 \oplus p_1$.)

   b. Conclude that $\{\bot, \top, \neg, \leftrightarrow, \oplus\}$ is not complete. (Hint: see the previous problem.)

∘ 8. How many ternary (3-ary) complete connectives are there?

∘ 9. Do problem 7 on page 28.

$\star$ 10. Do problem 8 on page 28. (Hint: it might help to read problem 7.)

11. Make up a truth table for a ternary connective, and then find a formula that represents it.

12. Do problems 9 and 10 on page 28.

13. Using the property $\varphi \vee (\psi \wedge \theta) \approx (\varphi \vee \psi) \wedge (\varphi \vee \theta)$, and the dual statement with $\wedge$ and $\vee$ switched, put

$$(p_1 \wedge p_2) \vee (q_1 \wedge q_2) \vee (r_1 \wedge r_2)$$

in conjunctive normal form. (Hint: try it with $(p_1 \wedge p_2) \vee (q_1 \wedge q_2)$ first.)

$\star$ 14. Do problem 1 on page 39 of van Dalen. Remember that we are taking $\varphi \leftrightarrow \psi$ to abbreviate $(\varphi \to \psi) \wedge (\psi \to \varphi)$.

Note that a parenthesis is missing at the end of part (f). The $\leftarrow$ directions of parts (d) and (e) are a little tricky, because they require the classical rule RAA.

15. Do problem 2 on page 39.

There is a parenthesis missing in part (b); it should read $[\varphi \to (\psi \to \sigma)] \leftrightarrow [\psi \to (\varphi \to \sigma)]$. Here the square brackets are only used to make the formula more readable; they are no different from parentheses.