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Ice-ray: a note on the generation of Chinese lattice designs

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Abstract. The conventions used to construct traditional Chinese ice-ray lattice designs are investigated. Parametric shape grammars are defined for the recursive generation of these patterns.

Introduction

In his classic monograph, *A Grammar of Chinese Lattice*, Daniel Sheets Dye sets out a catalogue of traditional Chinese lattices constructed between 1000 BC and 1900 AD. Most of these ornamental window and grille designs, such as those shown in figure 1, have a clearly observable periodic or regular structure that allows for their straightforward generation by simple shape grammars (Stiny, 1975; 1976). For example, the shape grammar specified in figure 2(a) generates the lattice shown in figure 1(b), as indicated in figure 2(b). (A formal definition of shape grammars of this type is given in the appendix. In this shape grammar, and in the shape grammar specified in figure 4, the 'bars' in ice-ray designs are represented as single straight lines.) The group of Chinese lattice called *ice-ray*, however, do not exhibit this periodicity or regularity. A representative sample of ice-ray lattices is shown in figure 3.

> "To appreciate [these] designs ... one needs to see ice forming on quiet water on a cold night. Straight lines meet longer lines, making unique and beautiful patterns. The Chinese term this *ice-line*, or lines formed by cracking ice; I have described it as the result of a molecular strain in shrinking or breaking, but more recent observations and photographs seem to prove that it is a conventionalization of ice-formation which has become traditional." (Dye, 1949, page 298)

The conventions used to construct Chinese ice-ray designs are investigated in this note. Parametric shape grammars are defined that generate ice-ray lattices in a very simple and intuitively compelling way. These shape grammars differ from standard shape grammars, such as the one in figure 2(a), in that they contain shape rules defined in terms of labelled *parameterized* shapes. Some familiarity with the definition and application of standard shape grammars is assumed in the following discussion. Readers new to shape grammars are referred to Stiny (1975; 1976).

Preliminary definitions

A *shape* is a finite arrangement of straight lines of limited but nonzero length in two or three dimensions. Shapes are specified by drawing them in a Cartesian coordinate system. This coordinate system is usually not given explicitly, its origin, axes, and units being understood. The shape containing no lines is called the *empty shape* and is denoted by $s_{\emptyset}$.

A shape s_1 is a *subshape* of a shape s_2 (denoted by $s_1 \mathrel{\hat{\subseteq}} s_2$) if and only if every part of s_1 is also a part of s_2. That is, s_1 coincides point for point with some part of s_2 in the coordinate system in which they are drawn.

The *shape union* of shapes s_1 and s_2 (denoted by $s_1 \mathbin{\hat{\cup}} s_2$) is the shape consisting of the lines in both s_1 and s_2. The shape union of shapes is specified by superimposing their drawings so that the coordinate systems in which they are drawn coincide.

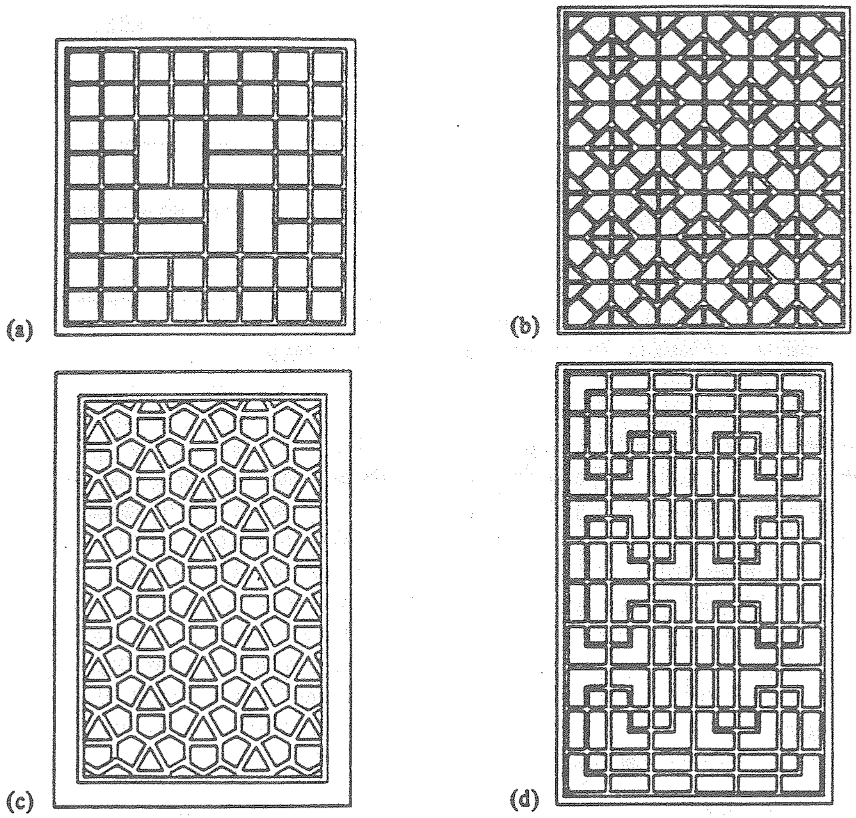


Figure 1. Chinese lattice designs: (a) near Kwangyüan, Szechwan, 1875 AD; (b) Chengtu, Szechwan, 1825 AD; (c) Chengtu, Szechwan, 1800 AD; (d) Hanchow, Szechwan, 1875 AD.

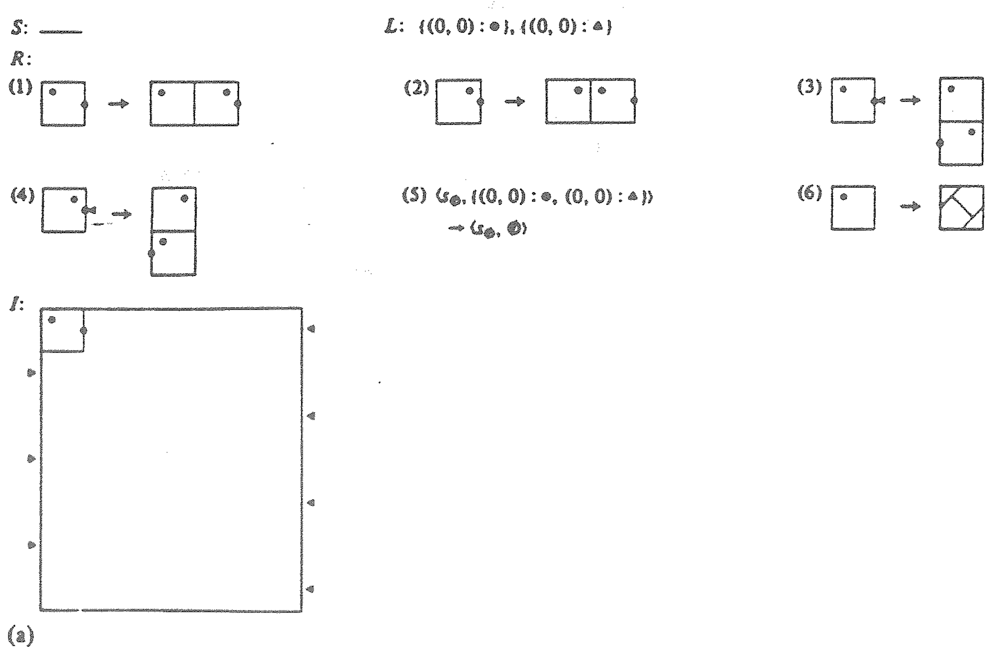


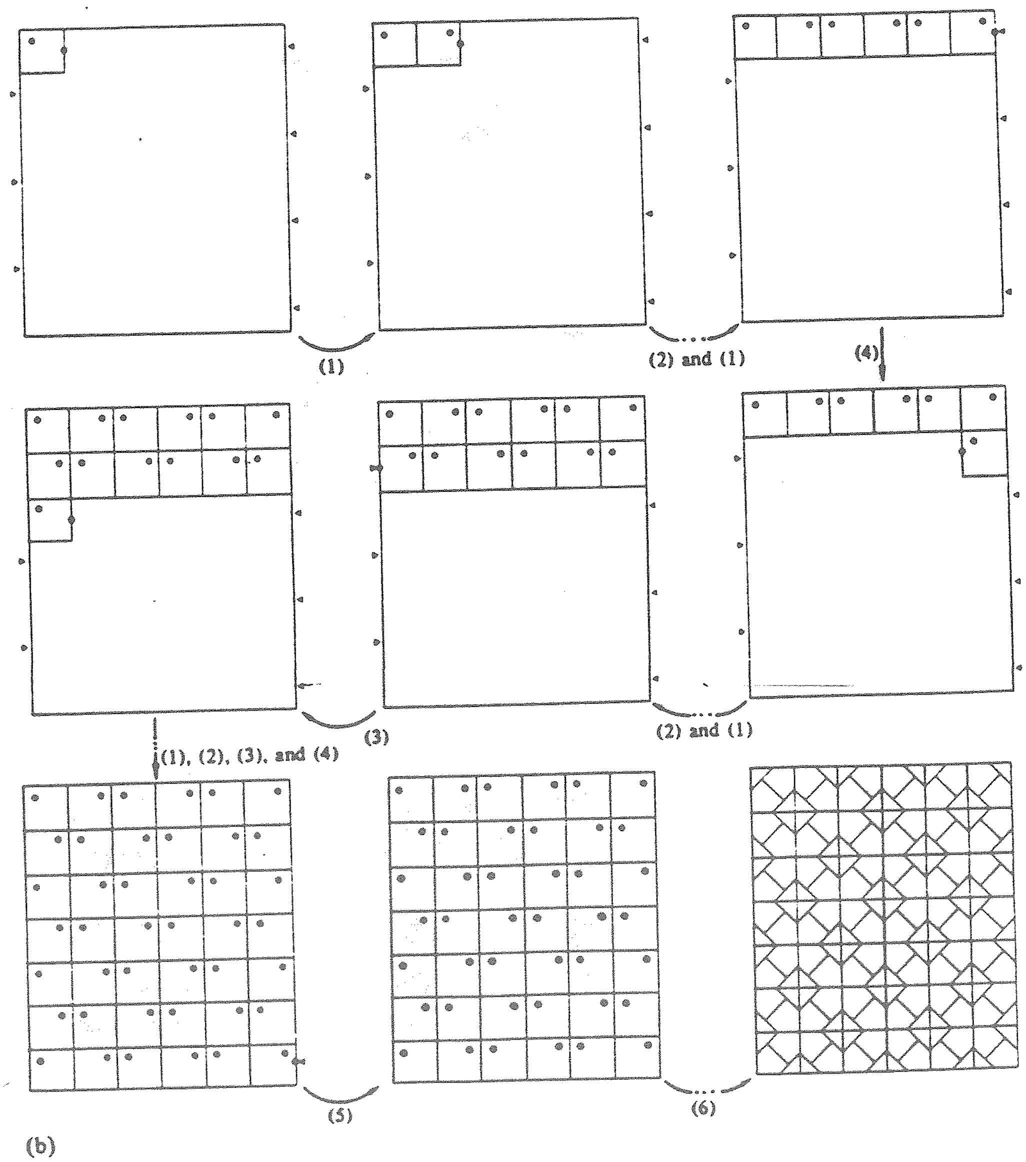
Figure 2. The shape grammar specified in (a) can be used to generate the lattice design shown in figure 1(b) as indicated in (b).

The *shape difference* of shapes s_1 and s_2 (denoted by $s_1 \dot{-} s_2$) is the shape obtained from s_1 by erasing that part of s_1 that coincides with s_2.

The *Euclidean transformations* are translation, rotation, scale, and mirror image, or finite compositions of these. A transformation τ of a shape s is the shape denoted by $\tau(s)$.

Given a finite set of shapes S, the set of shapes S^+ is the least set containing the shapes in S and which is closed under shape union and the Euclidean transformations. For example, if S contains a single shape consisting of a single straight line, then S^+ contains all of the rectilinear shapes. The set of shapes S^* is given by $S^* = S^+ \cup \{s_\emptyset\}$.

A family of shapes can be defined by associating parameters or parametric expressions satisfying certain conditions with a limited number of points coincident with lines in a given shape. A particular member of this family is specified by giving an *assignment* of real values to the parameters that satisfies the conditions. The result of applying an assignment g to a *parameterized shape* s is the shape denoted by $g(s)$.

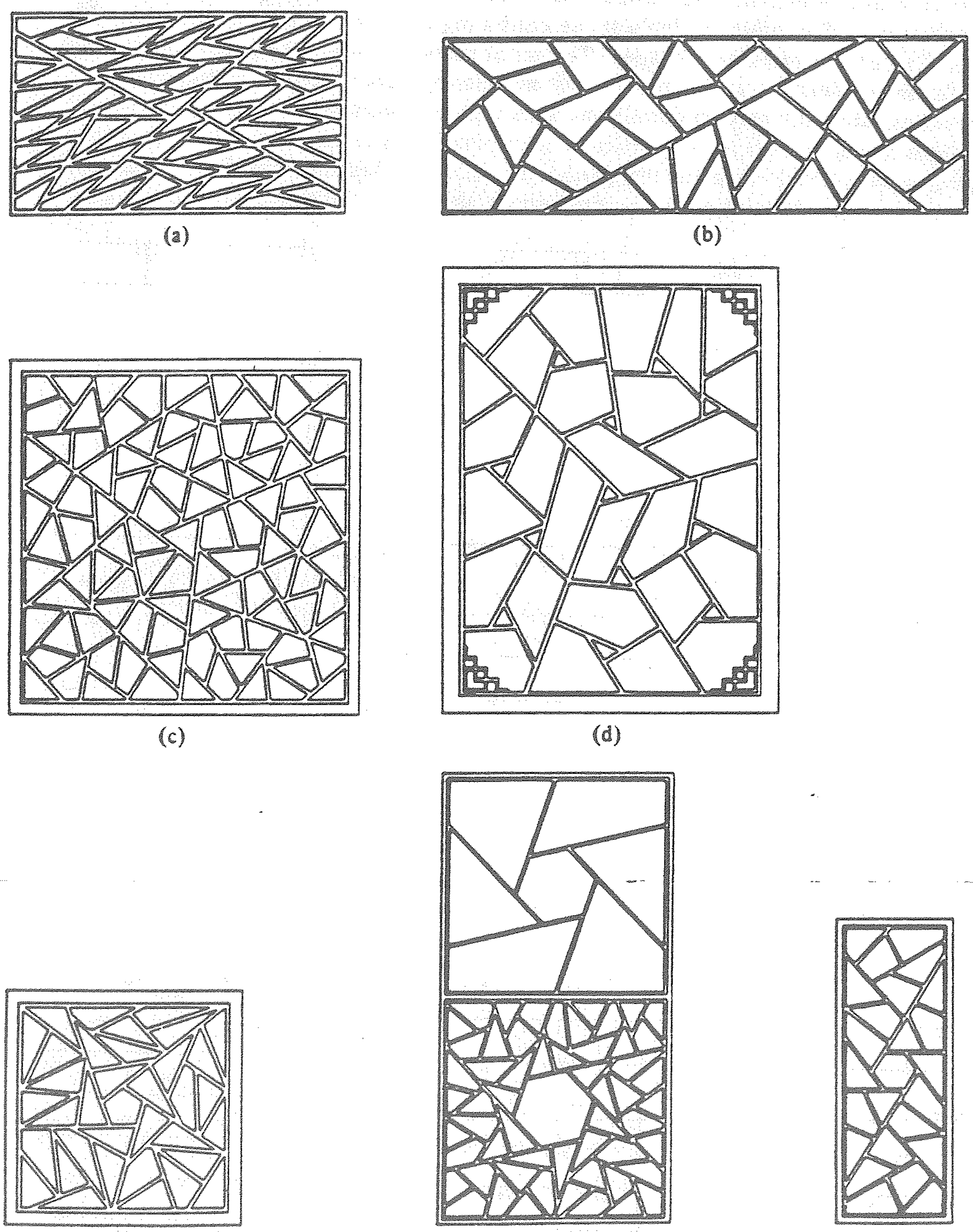


(b)

Figure 2 (continued)

A *labelled point* $p:A$ is a point p with a symbol A associated with it. Two labelled points $p_1:A_1$ and $p_2:A_2$ are equivalent if and only if $p_1 = p_2$ and A_1 is identical to A_2. A transformation τ of a labelled point $p:A$ is the labelled point $\tau(p):A$, where $\tau(p)$ is the image of p under τ.

An *unordered set* of labelled points is a finite collection of labelled points not all of which need be nonequivalent, such as $\{p_1:A_1, p_1:A_1, p_1:A_1, p_2:A_2, p_2:A_2, p_3:A_3\}$.



(a) (b) (c) (d) (c) (f) (g)

Figure 3. Ice-ray lattice designs: (a) Chengtu, Szechwan, 1850 AD; (b) Chengtu, Szechwan, 1800 AD; (c) Chengtu, Szechwan, 1880 AD; (d) Jungking, Szechwan, 1725 AD; (e) Chengtu, Szechwan, 1875 AD; (f) Kwanhsien, Szechwan, 1875 AD; (g) Chengtu, Szechwan, 1875 AD.

An unordered set of labelled points l_1 is a subset of an unordered set of labelled points l_2 (denoted by $l_1 \subseteq l_2$) if and only if for each occurrence of a labelled point in l_1 there is a corresponding occurrence of an equivalent labelled point in l_2. The union of unordered sets of labelled points l_1 and l_2 (denoted by $l_1 \cup l_2$) is the unordered set of labelled points containing each occurrence of a labelled point in l_1 and each occurrence of a labelled point in l_2. The difference of unordered sets of labelled points l_1 and l_2 (denoted by $l_1 - l_2$) is the unordered set of labelled points obtained from l_1 by deleting each occurrence of a labelled point in l_1 for which there is an occurrence of an equivalent labelled point in l_2. A transformation τ of an unordered set of labelled points l is the unordered set of labelled points $\tau(l)$ given by $\tau(l) = \{\tau(p) : A | p : A$ is an occurrence of a labelled point in $l\}$.

Given a finite set of unordered sets of labelled points L, the set of unordered sets of labelled points L^+ is the least set containing the elements in L and which is closed under union and the Euclidean transformations. The set of unordered sets of labelled points L^* is given by $L^* = L^+ \cup \{\emptyset\}$, where $\emptyset$ is the empty set.

A family of unordered sets of labelled points can be defined by associating parameters or parametric expressions satisfying certain conditions with the different points in a given unordered set of labelled points. A particular member of this family is specified by giving an assignment of real values to the parameters that satisfies the conditions. The result of applying an assignment g to an unordered set of labelled *parameterized points* l is the unordered set of labelled points denoted by $g(l)$.

A *labelled shape* consists of a shape and an unordered set of labelled points. More precisely, a labelled shape σ is given by an ordered pair $\sigma = \langle s, l \rangle$, where s is a shape and l is an unordered set of labelled points. The labelled points in l may be coincident with the lines in s, but this need not be the case. A labelled shape $\sigma = \langle s, l \rangle$ is specified by drawing s in a Cartesian coordinate system and placing the labels in l next to the points with which they are associated. A shape s is the labelled shape $\langle s, \emptyset \rangle$.

Relations and operations on shapes can be extended to labelled shapes in the obvious way:

For labelled shapes $\sigma_1 = \langle s_1, l_1 \rangle$ and $\sigma_2 = \langle s_2, l_2 \rangle$, σ_1 is a subshape of σ_2 (denoted by $\sigma_1 \dot{\subseteq} \sigma_2$) if and only if $s_1 \dot{\subseteq} s_2$ and $l_1 \subseteq l_2$.

The shape union of σ_1 and σ_2 (denoted by $\sigma_1 \dot{\cup} \sigma_2$) is the labelled shape $\langle s_1 \dot{\cup} s_2, l_1 \cup l_2 \rangle$.

The shape difference of σ_1 and σ_2 (denoted by $\sigma_1 \dot{-} \sigma_2$) is the labelled shape $\langle s_1 \dot{-} s_2, l_1 - l_2 \rangle$.

A transformation τ of a labelled shape $\sigma = \langle s, l \rangle$ is the labelled shape $\tau(\sigma)$ given by $\tau(\sigma) = \langle \tau(s), \tau(l) \rangle$.

A *labelled parameterized shape* σ is given by $\sigma = \langle s, l \rangle$, where s is a parameterized shape and l is an unordered set of labelled parameterized points. An assignment g to the parameters in s and l specifies a particular labelled shape $g(\sigma) = \langle g(s), g(l) \rangle$ in the family of labelled shapes defined by σ.

Parametric shape grammars

A parametric *shape grammar* has five parts:

(1) S is a finite set of shapes.

(2) L is a finite set of unordered sets of labelled points.

(3) R is a finite set of *shape rules* of the form $\alpha \rightarrow \beta$, where α and β are labelled parameterized shapes: $\alpha = \langle u, i \rangle$, and $\beta = \langle v, j \rangle$. Any assignment g to the parameters in the parameterized shapes u and v, and the unordered sets of labelled parameterized points i and j, results in shapes $g(u)$ and $g(v)$, that are in S^*, and unordered sets of labelled points $g(i)$ and $g(j)$, that are in L^+ and L^* respectively.

(4) I is a labelled shape such that $I = \langle w, k \rangle$, where w is a shape in S^{*}, and k is an unordered set of labelled points in L^{+}. The labelled shape I is called the *initial shape*.

(5) T is a set of transformations.

A shape is generated by a shape grammar by beginning with the initial shape I and recursively applying the shape rules in the set R. A shape rule $\alpha \rightarrow \beta$ applies to a labelled shape γ when there is an assignment g and a transformation τ such that

$$\tau(g(\alpha)) \hat{\subseteq} \gamma \, .$$

The result of applying the shape rule $\alpha \rightarrow \beta$ to the labelled shape γ under g and τ is another labelled shape given by

$$(\gamma \dot{-} \tau(g(\alpha))) \hat{\cup} \tau(g(\beta)) \, .$$

The shape generation process terminates when no shape rule in the set R can be applied. The *language* defined by a shape grammar is the set of shapes s generated by the shape grammar, that is, labelled shapes of the form $\langle s, \emptyset \rangle$.

The relationship between this definition of parametric shape grammars and the definition of standard shape grammars is discussed in the appendix.

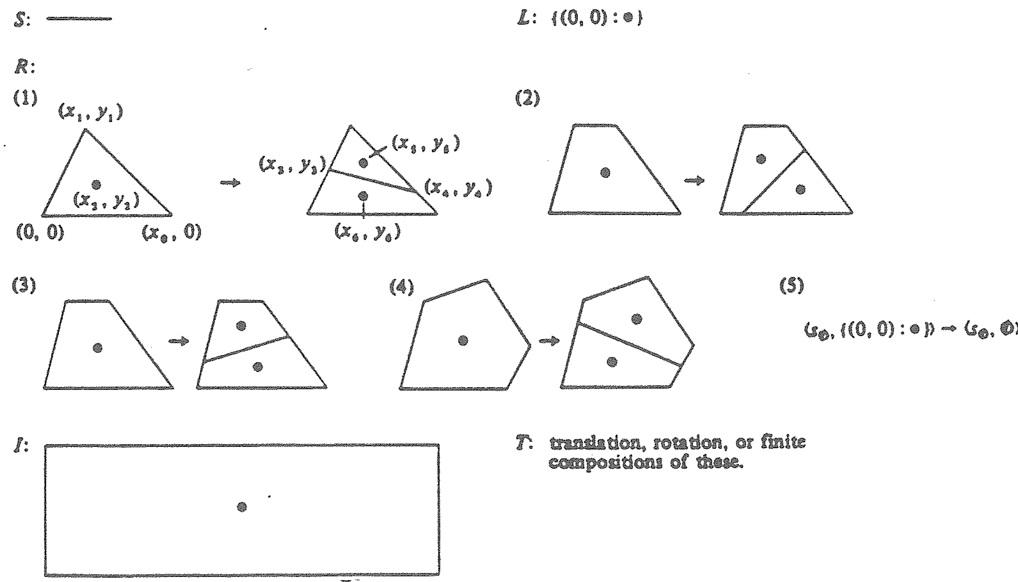


Figure 4. A shape grammar that generates the ice-ray design shown in figure 3(b).

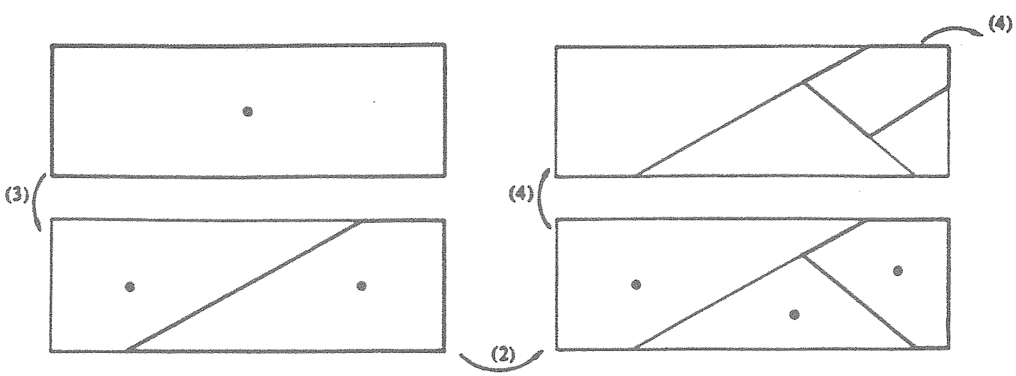


Figure 5. A generation of the ice-ray design shown in figure 3(b) by means of the shape grammar specified in figure 4. The labels • are omitted in steps 3 et seq.

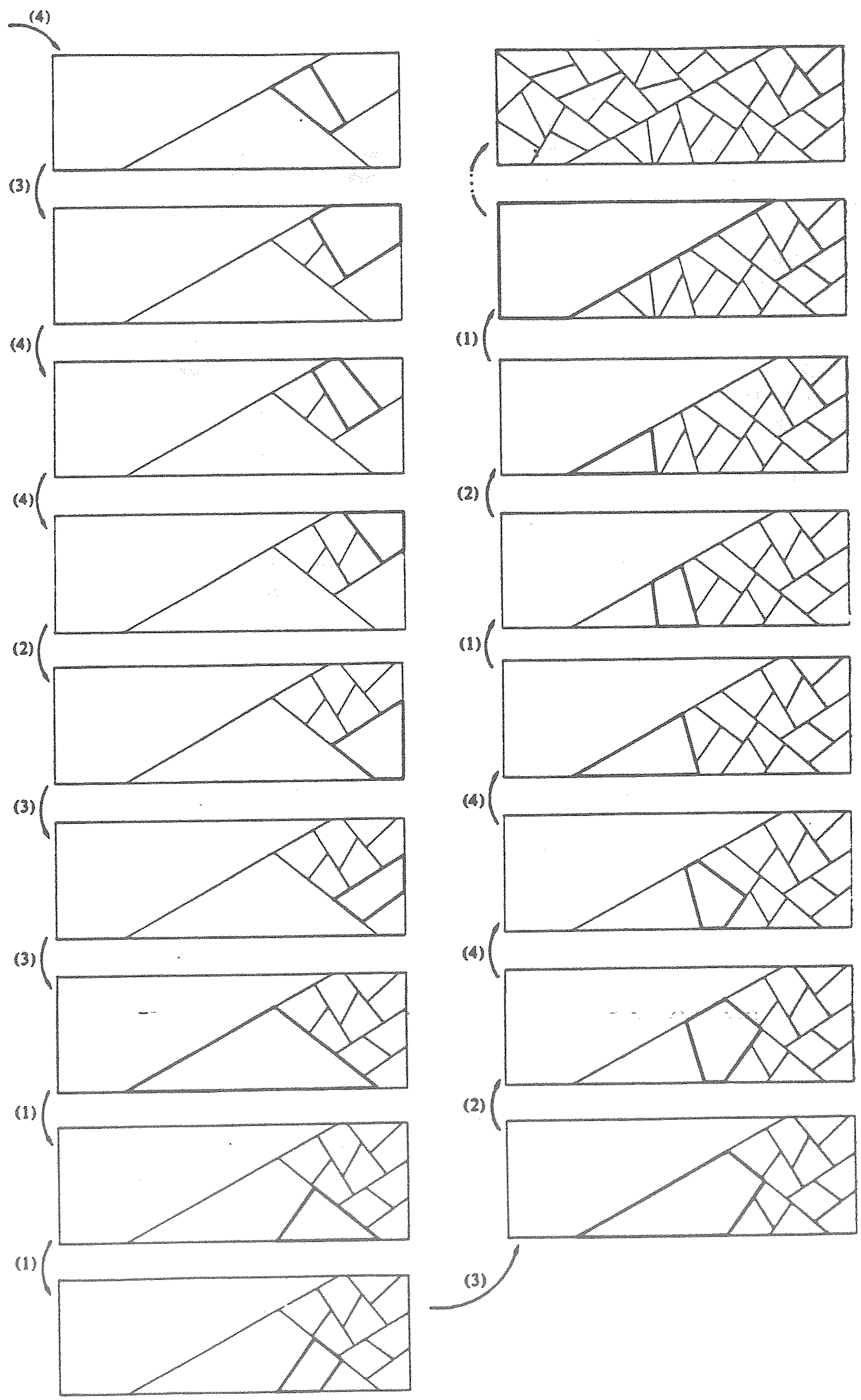


Figure 5 (continued)

Shape grammars that generate Chinese ice-ray lattices

The shape grammar specified in figure 4 generates the ice-ray design shown in figure 3(b). This shape grammar has five shape rules.

The first shape rule states that any triangle with area greater than some given constant may be augmented once by placing a line between any two of its edges to form another triangle and a quadrilateral with approximately equal areas. More precisely, the left side of this shape rule consists of a parameterized triangle with vertices at the points (0, 0), $(x_0, 0)$, and (x_1, y_1), and the symbol ● associated with the parameterized point (x_2, y_2). The parameters x_0, x_1, y_1, x_2, and y_2 satisfy the following conditions:

(1) The area of the triangle with vertices at (0, 0), $(x_0, 0)$, and (x_1, y_1) is greater than some specified constant c.

(2) The point (x_2, y_2) is at a distance from the centroid of the triangle, with vertices at (0, 0), $(x_0, 0)$, and (x_1, y_1), equal to rA_1/A_2, where r is the radius of the greatest circle contained in the triangle centred on the centroid, A_1 is the area of this circle, and A_2 is the area of the triangle.

Condition (1) prevents the application of the shape rule to triangles that are too small. Notice that shape rule application would not be restricted in this way if the set of transformations T contained the scale transformation. The purpose of condition (2) is explained below.

The right side of the first shape rule consists of the parameterized triangle in its left side with a line drawn between two of its edges to form another triangle and a quadrilateral. One end point of this line is located at (x_3, y_3) and the other end point at (x_4, y_4). The symbol ● is associated with the parameterized points (x_5, y_5) and (x_6, y_6). The parameters x_3, y_3, x_4, y_4, x_5, y_5, x_6 and y_6 satisfy the following conditions:

(3) The point (x_3, y_3) is coincident with the line having end points (0, 0) and (x_1, y_1), but not with these end points.

(4) The point (x_4, y_4) is coincident with the line having end points $(x_0, 0)$ and (x_1, y_1), but not with these end points.

(5) The absolute value of the difference between the areas of the triangle with vertices (x_3, y_3), (x_4, y_4), and (x_1, y_1) and the quadrilateral with vertices (0, 0), $(x_0, 0)$, (x_4, y_4), and (x_3, y_3) is less than some specified constant d.

(6) The point (x_5, y_5) is defined for the triangle with vertices (x_3, y_3), (x_4, y_4), and (x_1, y_1) as in rule (2).

(7) The point (x_6, y_6) is defined for the quadrilateral with vertices (0, 0), $(x_0, 0)$, (x_4, y_4), and (x_3, y_3) similar to rule (2).

The usual algebraic and analytic expressions can be given for these conditions. Condition (5) is used to ensure that the areas of the triangle and quadrilateral produced by the addition of the line with end points (x_3, y_3) and (x_4, y_4) are approximately equal. Conditions (1) and (5) together with the dimensions of the initial shape determine the maximum number of times the first shape rule can be applied in the generation of a shape. Conditions (2), (6), and (7) determine the positions of the symbol ● in the left and right sides of this shape rule. These positions prevent the shape rule from applying to the same triangle more than once. Because this shape rule and the other shape rules in the shape grammar apply under the Euclidean transformations of translation, rotation, or finite compositions of these (that is, transformations in the set T), it is not necessary to give general parameterizations for the shapes and labelled points occurring in them.

The second, third, and fourth shape rules of this shape grammar have the same general properties as the first shape rule. The second and third shape rules state that any convex quadrilateral with area greater than some given constant can be augmented once by (a) placing a line between any two of its adjacent edges to form a triangle and a convex pentagon with approximately equal areas or (b) placing a line between any two of its nonadjacent edges to form two additional convex quadrilaterals with approximately equal areas. The fourth shape rule states that any convex pentagon with area greater than some given constant can be augmented once by placing a line between any two of its nonadjacent edges to form a convex quadrilateral and another convex pentagon with approximately equal areas.

The fifth shape rule in this shape grammar allows for the symbol ● to be erased, and hence for the termination of the shape generation process. The left side of this shape rule consists of the empty shape and the symbol ● associated with the point (0, 0); the right side consists of the empty shape only. Because shape rules apply under the transformations in T, in particular, under translation, it is not necessary to associate ● with a parameterized point.

Figure 5 shows the generation of the ice-ray lattice shown in figure 3(b) by making use of the shape grammar specified in figure 4. At each step in the generation, the indicated shape rule applies to the polygon with the bold outline.

The ice-ray design shown in figure 3(a) can be generated by using the shape rules in the shape grammar of figure 4, beginning with the outside rectangle in the design. The ice-ray designs in figure 3(c)–(g) can be generated by using shape rules like these and some additional shape rules defined to allow polygons to be augmented by inscribing the triangular, pentagonal, or hexagonal shapes shown in figure 6. The reader is invited to specify these simple shape grammars.

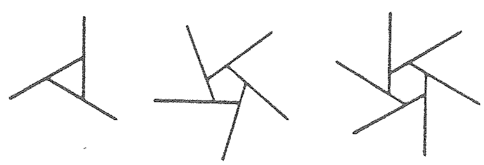


Figure 6. Shapes commonly inscribed in polygons to make ice-ray designs.

Discussion

> "In the case of the ice-ray pattern, [the artisan] divides the whole area into large and equal light spots, and then subdivides until he reaches the size desired; he seldom uses dividers in his work." (Dye, 1949, page 17)

One can imagine a Chinese artisan, summoned to a building site, bringing with him tools and implements and a collection of finely finished sticks. Shown a rectangular window frame, he is asked to create an ice-ray lattice. He begins his design by selecting a stick of the appropriate length and carefully attaching it between two edges of the existing rectangular frame, thus forming two quadrilateral regions (shape rule 3). He continues his work by subdividing one of these areas into a triangle and a pentagon (shape rule 2). He further divides the triangle into a triangle and a quadrilateral (shape rule 1); he divides the pentagon into a quadrilateral and a pentagon (shape rule 4). Each subdivision is made in the same way: attach an appropriately sized stick between two edges of a previously constructed triangle or quadrilateral or pentagon so that it does not cross previously inserted pieces. Each stage of the construction is stable; each stage follows the same rules. Indeed, the steps in the ice-ray lattice generation given in figure 5 could well comprise the frames in a motion picture of the artisan creating his design!

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APPENDIX

The definition of standard shape grammars is obtained from the definition of parametric shape grammars by deleting part (5) and replacing part (3) with this statement: R is a finite set of *shape rules* of the form $\alpha \rightarrow \beta$, where α and β are labelled shapes; $\alpha = \langle u, i \rangle$ and $\beta = \langle v, j \rangle$. The shapes u and v are in S°; the unordered sets of labelled points i and j are in L^{+} and L° respectively.

If this definition of shape grammars is used, a shape rule $\alpha \rightarrow \beta$ applies to a labelled shape γ when there is a Euclidean transformation τ such that $\tau(\alpha) \mathrel{\hat{\subseteq}} \gamma$. The result of applying $\alpha \rightarrow \beta$ to γ under τ is the labelled shape $(\gamma \mathbin{\hat{-}} \tau(\alpha)) \mathbin{\hat{\cup}} \tau(\beta)$.

Notice that this definition of shape grammars uses labelled points instead of *markers*, as in the definitions of shape grammars given in Stiny (1975; 1976). Labelled points function in the same way as markers to guide the shape generation process. Labels, however, are invariant under the Euclidean transformations whereas markers are not.