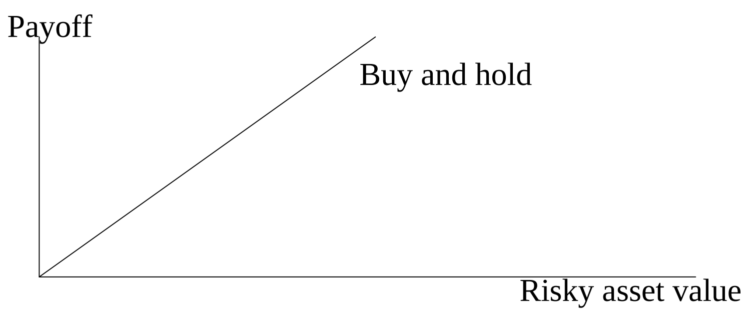
Objectives

- Recap of trading strategies
- Explain *why* mean-variance makes sense
- *compute* mean and standard deviation of portfolios
  - one risky asset and risk less
  - multiple risky assets
- Begin discussion of how to implement

Trading strategies

- Basic point: dynamic trading allows you to generate interesting payoff patterns



Which assets

- Portfolio insurance strategy:
  - example so far, stocks and risk free bonds
  - what about
    - domestic and international stocks
    - short term and long term bonds
    - market portfolio and sector bets
    - etc.

Mean-Variance Analysis Why?

- over 90% variation in performance measured by policy weights
- Investor preferences
  - like high expected returns
  - hate variance of portfolio
  - don't care about anything else
  - single period objective

Formulas

$$E[\tilde{r}] = \sum_{i=1}^{S} i \times \Pr(\tilde{r} = i)$$

$$\sigma_r^2 = Var[\tilde{r}] = E\left[(\tilde{r} - E[\tilde{r}])^2\right] = \sum_{i=1}^{S} (i - E[\tilde{r}])^2 \times \Pr(\tilde{r} = i)$$

$$\sigma_{x,y} = Cov(\tilde{x}, \tilde{y}) = E\left[(\tilde{x} - E[\tilde{x}])(\tilde{y} - E[\tilde{y}])\right]$$

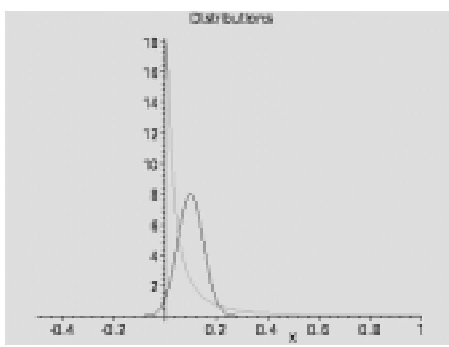
When does this make sense?

- Normal distributions of returns
  - only mean and variance determine shape of distribution
- Care about deviations from the mean equally...

When doesn’t it make sense?

- Skewed return distributions
- Dynamics Issues and dynamic trading...

Normal vs. Skewed Distribution



Portfolio

- Invest ω_i in asset I
- Total of N assets
- Mean and variance of portfolio?

$$E[\tilde{r}_p] = \omega_1 E[\tilde{r}_1] + \omega_2 E[\tilde{r}_2] + \cdots \omega_N E[\tilde{r}_N]$$

$$\sigma_{r_p}^2 = \sum_{i=1}^{N} \omega_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i} \omega_i \omega_j \sigma_{ij}$$

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Naïve Diversification

- Invest in N assets
- Invest 1/N in each asset
  - equally weighted portfolio

$$\sigma_p^2 = \sum_{i=1}^{N} \frac{1}{N^2} \sigma_i^2 + \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{N^2} \sigma_{ij}$$
$$= \frac{1}{N} AVERAGE\left(\sigma^2\right) + \frac{N-1}{N} AVERAGE\left(\sigma_{ij}\right)$$

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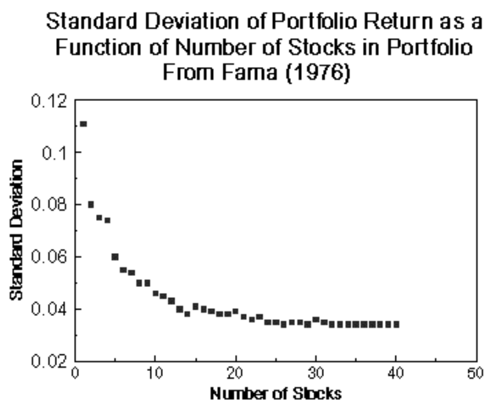
Naïve Diversification



One risky and riskless asset

- Invest ω in risky and (1- ω) in riskless
- Expected Return of Portfolio?

$$E\left[\tilde{r}_p\right] = \omega E\left[\tilde{r}\right] + (1-\omega) r_f$$
$$= r_f + \omega\left(E\left[\tilde{r}\right] - r_f\right)$$

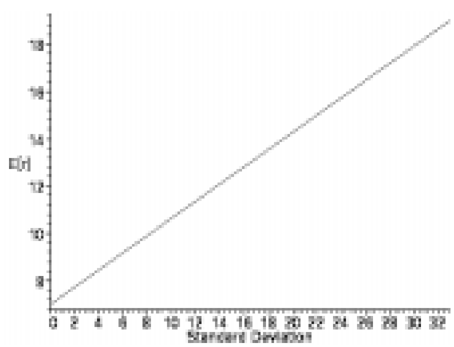
•Portfolio Standard Deviation

$$\sigma_p = \sqrt{\omega^2 \sigma^2}$$
$$= \omega\sigma$$

Example

- Risk free rate: 7%
- Risky asset
  – expected return: 15%
  – standard deviation: 22%
- Premium on risky asset: 8%
- Invest half in each
  – expected return: 11%, standard dev: 11%

Varying Investment Proportions



Punchline

- Linear relationship between expected return and standard deviation
- Capital Allocation Line (CAL)
- slope: extra reward for risk

$$\sigma_p = \omega\sigma \rightarrow \omega = \frac{\sigma_p}{\sigma}$$
$$E[\tilde{r}_p] = r_f + \omega\left(E[\tilde{r}] - r_f\right)$$
$$E\left[\tilde{r}_p\right] = r_f + \frac{\left(E[\tilde{r}] - r_f\right)}{\sigma}\sigma_p$$

Which risky asset to choose?

- combine risk free with some index
  - easy to do, low transactions costs
  - CML
- If you could choose one risky , want highest slope, or...

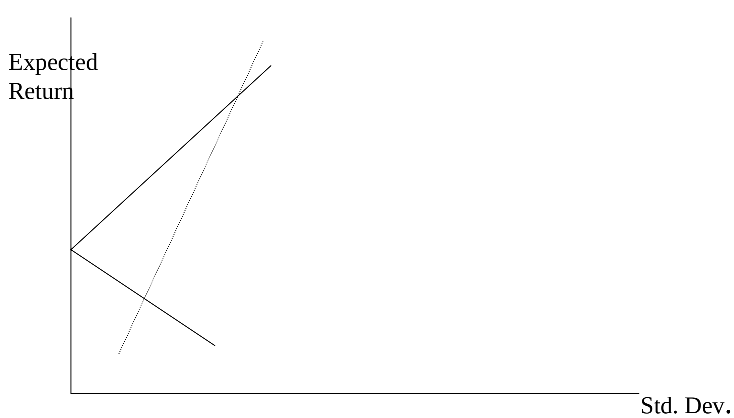
$$Max_{\tilde{r}_{port}} \frac{\left(E\left[\tilde{r}_{port}\right] - r_f\right)}{\sigma_{port}}$$

Two risky assets

- Invest ω in asset 1 and (1- ω) asset 2

$$E[\tilde{r}_p] = \omega E[\tilde{r}_1] + (1-\omega)[\tilde{r}_2]$$
$$\sigma_p^2 = \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\sigma_{12}$$
$$= \omega^2\sigma_1^2 + (1-\omega)^2\sigma_2^2 + 2\omega(1-\omega)\rho_{12}\sigma_1\sigma_2$$

Diversification and Correlation



So far...

- Combining riskless and one risky
  - straight line
  - choosing risky: maximize slope
- Multiple risky assets
  - two assets, correlation coefficient is key
  - with lots of assets, average covariance matters

Multiple Risky Assets

- Vector notation
- ω: N by 1 vector of asset weights
- Σ: variance-covariance matrix of asset returns, N by N matrix
- μ: N by 1 vector of expected returns
- ι: N by 1 vector of ones

Portfolios of Risky Assets

•Portfolio Expected Return:

$$E\left[\tilde{r}_p\right] = \omega' \mu$$

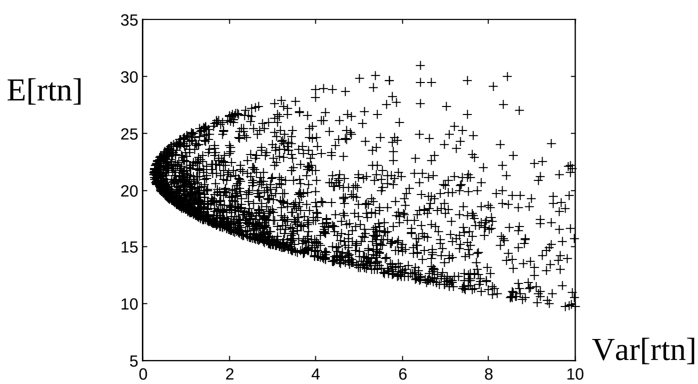
•Portfolio Variance:

$$\sigma_p^2 = \omega' \Sigma \omega$$

•Constraint

$$1 = \omega' \iota$$

Feasible Portfolios



Frontier of Risky Assets

- Minimize portfolio variance subject to
  - mean constraint
    - changing constraint traces out entire curve
  - portfolio constraint

$$
\begin{aligned}
&Min_{\omega}\sigma_m^2 = \omega' \Sigma \omega \\
&st. \\
&\omega' \mu = m \\
&\omega' \iota = 1
\end{aligned}
$$

Properties of Solution

- Can be done in Excel using solver

- First Order Conditions:

$$
\begin{aligned}
&\Sigma^{-1}\omega^* = 0.5\lambda_1\mu + 0.5\lambda_2\iota \\
&\omega^{*}{}' \mu = m \\
&\omega^{*}{}' \iota = 1
\end{aligned}
$$

Two Fund Separation

- Useful to compute results
- **All** solutions to the problem are a combination of two portfolios
  - minimum variance portfolio
  - arbitrary optimal portfolio

Minimum Variance Portfolio

- Lowest possible portfolio variance
- Ignore mean constraint
- Use solver in Excel

$$Min_{\omega}\sigma^{2}_{mv} = \omega`\Sigma\omega$$
$$s.t.$$
$$\omega`\iota = 1$$

How to generate frontier

- Solve for global minimum variance portfolio: r_{mv}
  - compute the mean, m_{mv}
- Pick arbitrary mean, and solve for optimal portfolio at that mean: r_m
- Any other mean, mn
  - weight w, st. w m_{mv}+(1-w)m=mn

Example

- 3 assets
- global minimum variance portfolio
  - [.2,.5,.3]
  - mean at this point 4%
- efficient portfolio with mean of 10%
  - [.5,.1,.4]

Want optimal portfolio w/ mean of 7%?

Summary

- Mean Variance Analysis
- why/why not
- two asset portfolios
- role of covariance
- two fund result

Next Time

- Add investment in riskless asset…
- Excel example?
- Practical issues
- Equilibrium: The CAPM and APT
- References
  - Chapters 9, 10 in text
  - Kritzman on Factor models.