Introduction to Bond Risk Measures

Last Time

- Term Structure: spot and forward rates
- Arbitrage restrictions

Today

- Yield to maturity
- Duration: what and why

Solving for Term Structure

Basic Idea

- No arbitrage $\Rightarrow$ consistent PDB prices
- For all bonds

$$P = \sum_i cf_i b_i$$

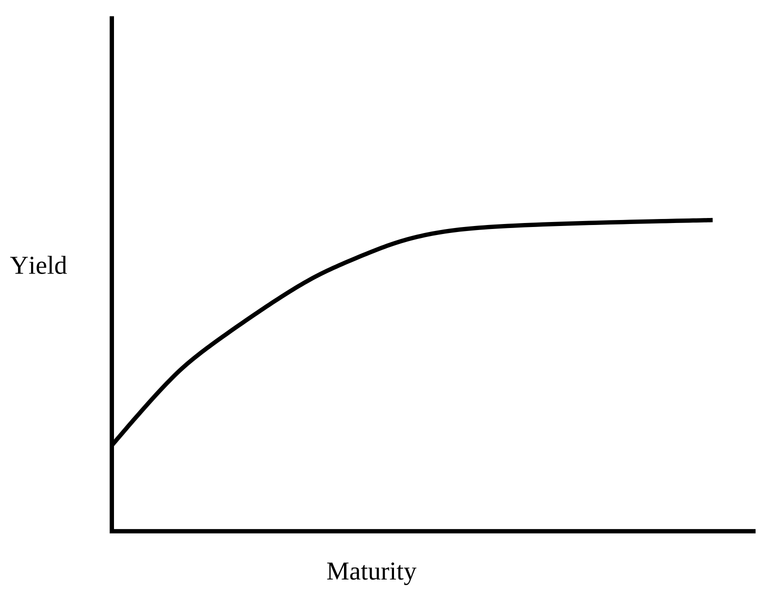
$$b_i = 1/(r_i)^i,\ 0 < b_i \leq 1$$

$$b_i \geq b_{i+1}$$

$$P_i = cf_1 b_1 + cf_2 b_2 + ... + cf_T b_T$$

- T unknowns
- $\rightarrow$ need T independent equations to get unique solutions
- $< T$ independent equations $\rightarrow$ can't solve uniquely for all b_i
- $> T$ independent equations $\rightarrow$ overidentified

Example of Term Structure



Spot and Forward Rates

Spot rates are average of forward rates:

$$r_t^t = r_{t-1}^{t-1} \; {}_{t-1}f_t = r_{t-2}^{t-2} \; {}_{t-2}f_{t-1} \; {}_{t-1}f_t$$

$$r_t^t = \prod_{j=1}^{t} {}_{j-1}f_j =_0 f_1 \; {}_1f_2 \ldots {}_{t-1}f_t$$

$$\Rightarrow \; r_t = \left(\prod_{j=1}^{t} {}_{j-1}f_j \right)^{\left(\frac{1}{t}\right)}$$

1. TS increasing ${}_{t-1}f_t > r_t$
2. TS decreasing ${}_{t-1}f_t < r_t$
3. TS flat ${}_{t-1}f_t = r_t$

Yield to Maturity

Yield: IRR of the bond, or interest rate that equates price and discounted cash flow.

1. P bond price
2. C coupon
3. F face
4. T time to maturity

$$P = \frac{C}{Y} + \frac{C}{Y^2} + \ldots + \frac{F + C}{Y^T}$$

Yield is *DERIVED* from prices!

- $Y = c$ then $P = F$ trading at **PAR**
- $Y > c$ then $P < F$ trading at **discount**
- $Y < c$ then $P > F$ trading at **premium**

Issues with Yields

- Basic tool in bond manager's toolbox
- Should you buy high yield bonds and sell low yield bonds?
- These are riskless flows — does risk matter?
- Yield doesn't say much, unless term structure flat
- Spots vs. Yields

- **NPV**: discount cash flows at the same time at the same rate, and cash flows at different times differently.
  Yield: discounts cash flows differently at the same time, the same at different times.

$$P_A = 10b_1 + 110b_2 = \frac{10}{Y_A} + \frac{110}{Y_A^2}$$

$$P_B = 6b_1 + 6b_2 + 106b_3 = \frac{6}{Y_B} + \frac{6}{Y_B^2} + \frac{106}{Y_B^3}$$

- Two bonds with same maturity and different coupons with different yields:
  $r_1 = 1.1$, $r_2^2 = 1.32$, $r_2 = 1.148$
  Bond A 2yr, 10% coupon, 100 F, Bond B 2yr, 5%, 100 F

$$P_A = \frac{10}{1.1} + \frac{110}{1.32} = 92.42$$

$$P_B = \frac{5}{1.1} + \frac{105}{1.32} = 84.0909$$

$$Y_A : 92.42 = \frac{10}{Y_A} + \frac{110}{Y_A^2} \Longrightarrow Y_A = 1.1464$$

$$Y_B : 84.0909 = \frac{5}{Y_B} + \frac{105}{Y_B^2} \Longrightarrow Y_B = 1.1476$$

NOT THE SAME

- **Yield** assumes you can reinvest at Y

  $r_1 = 1.1$, $r_2^2 = 1.32$, $r_2 = 1.148$
  ${}_1f_2 = \frac{1.32}{1.1} = 1.2$
  Bond A 2yr, 10% coupon, 100 F

$$\begin{aligned} P_A &= \frac{10}{1.1} + \frac{110}{1.32} \\ &= \frac{10}{1.1} + \frac{110}{(1.1)({}_1f_2)} \\ &= \frac{10}{1.1} + \frac{110}{(1.1)(1.2)} \\ &= 92.42 \end{aligned}$$

Duration

- Effective maturity of the bond
- PDB: maturity # yrs. until face paid
- Coupon Bonds: lots of payment dates
- **Duration** or effective maturity: average time until you get paid

- Cash Flows: cf_i
- price: B
- Duration: (Macaulay/ Fisher–Weil)

$$D = \sum_{i=1}^{N} \frac{P.V.\ cf_i}{B} \times i$$

weighted average of maturity of each cash flow, weights are PV of the cash flows

- Macaulay also came up with **Simple Duration**: use YTM instead of r_i

$$D^s = \sum_{i=1}^{N} \frac{cf_i (Y)^{-i}}{B} \times i$$

- $D = D^s$ iff $r_1 = r_2 = ... = r_n = Y$
- If term structure not flat, use D, not D_s

Example

$$r_1 = r_2 = 1.08$$

2 bonds with maturity 5 years, $F = 1000$
Bond A — coupon of 4%
Bond B — coupon 8%.

BOND A

Year	cf_i	b_i	PV of cf_i	PV as a % of Price	(1) × (5)
1	40	.9259	37.04	.0441	.0441
2	40	.8573	34.29	.0408	.0816
3	40	.7938	31.75	.0378	.1134
4	40	.7350	29.40	.0350	.1400
5	1040	.6806	707.82	.8423	4.2115
			840.30	1.0000	4.5906

Duration of Bond A = 4.59 years

BOND B

Year	cf_i	b_i	PV of cf_i	PV as a % of Price	(1) $\times$ (5)
1	80	.9259	74.07	.0741	.0741
2	80	.8573	68.59	.0686	.1372
3	80	.7938	63.50	.0635	.1905
4	80	.7350	58.80	.0588	.2352
5	1080	.6806	735.04	0.7350	3.6750
			1000.00	1.0000	4.3120

Duration of Bond B = 4.31 years

Points

- duration $\leq$ bond maturity
- more the bond's value from earlier cash flows, lower duration, etc.

Fulcrum Interpretation

Application: Increasing Yields

Coupon Payments

Duration vs. Time

Risk Measure Interpretation

For **parallel shifts** of the yield curve:

$$\frac{\partial B}{B} = -D\frac{\partial Y}{Y}$$

$$\frac{\partial B}{\partial Y} \times \frac{Y}{B} = -D$$

Examples

- Assume $y = 1.08$
- 2 bonds: 30 year PDB and 1 year PDB
- durations?
- $b_1 = \frac{1}{1.08} = 0.93$, $b_{30} = 0.0994$

- yield rises to 1.082
- % Change
  - new price:$b_1 = \frac{1}{1.082}$, or 0.185% drop
  - new price $b_{30} = \frac{1}{1.082^5} = 0.094$, or 5.4% drop

Duration Approximation

Example from earlier:

$$D = 4.49$$

$$r = 1.08$$

$$B = 840.30$$

Rate rises to 1.082
Duration Approx:

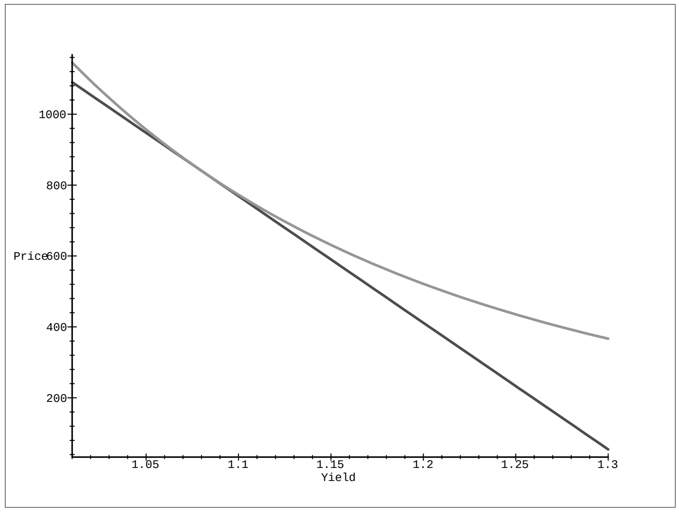
$$\frac{\Delta r}{r} = \frac{1.082 - 1.08}{1.08} = 0.00185$$

$$\frac{\Delta B}{B} \approx -D\frac{\Delta r}{r} = -0.00849$$

Actual Change

$$-0.00847$$

Bond Price vs. Yield



Key Points

- duration: linear approximation to price function
- only good for small changes
- only good for parallel shifts of yield curve

Portfolio Duration

- You invest w_i % of your portfolio in bond i
- D_i duration of bond i
- Portfolio Duration

$$D_p = w_1 D_1 + w_2 D_2 + ... + w_I D_I$$

- Intuition:

Example

- invest half your money in 1 and 2 year PDB
- Duration of portfolio?

Immunization

- Basic Idea: match duration of liabilities and funding
- Why?

Summary

- Yields Calculations
- Duration: What and why?

Next Time

- Other risk measures and applications
- Convexity
- Fixed, Floating Rate and Swaps
- Readings: Kritzman on Duration and Convexity, Sections 16.3 to end of Chapter 16