Administrative Issues

- HW # 1 due March 30
- next class: start at 2:00?
- TA Hours: 7-8 Wednesdays
  - This week: table by Cooper
  - After break: room 146
  - Same TA as Int. Finance (5:30-7)

Bond Arbitrage

Last Class

- replication of coupon bonds

Goals Today

- Spot and Forward Rates
- Restrictions on forward/spot rates /PDB prices
- Review of Yield to Maturity

Review of Bond Prices

- Pure discount bonds
- No Arbitrage
  - Price of coupon bond = price of replicating portfolio of PDBs

Bond Prices and Interest Rates

b_1 price of 1 period PDB
Define $r_1 \equiv 1$ year interest rate

$$b_1 = \frac{1}{r_1} \Longrightarrow r_1 = \frac{1}{b_1}$$

$r_i \equiv i$ year interest rate

$$b_i = \frac{1}{r_i^i} \Longrightarrow r_i = \left(\frac{1}{b_i}\right)^{1/i}$$

Generally,$r_i \neq r_j \Longrightarrow$ **TERM STRUCTURE**

Continuous Compounding

$$b_i = \exp(-ir_i)$$
$$\Longrightarrow r_i = -\frac{\log(b_i)}{i}$$

bond prices $\Longrightarrow$ term structure (spot interest rates)
Replication argument:

$$P = \sum_{i=1}^{N} b_i cf_i,$$

or,

$$P = \sum_{i=1}^{N} \frac{cf_i}{r_i^i},$$

Forward Rates

r_i today's rates on i period PDB's. Want to arrange borrowing/lending in the *future*.
e.g. meet today to arrange for advance borrowing 100$ in 1 yr. to be repaid in 2 yrs. **Forward Contract**

$${}_1f_2$$

Time:	0	1	2
Cash:	0	-100	$100{}_1f_2$

Rate fixed today: NO RISK

Forwards and Spots

Replication again!

1. buy a 2 year PDB. Get $\$100r_2^2$ in two years.
2. buy a 1 year PDB and forward contract

Time	0	1	2
(1) 1 yr PDB	-100	$100r_1$	0
(2) forward	0	$-100r_1$	$100r_1\ {}_1f_2$
Net	-100	0	$100r_1\ {}_1f_2$
(1) 2 yr PDB	-100	0	$100r_2^2$

NO ARBITRAGE $\Longrightarrow r_1 \, {}_1f_2 = r_2^2$,

$${}_1f_2 = \frac{r_2^2}{r_1}$$

r_i and b_i are related $b_i = \frac{1}{r_i^i}$

$$\Longrightarrow \; {}_1f_2 = \frac{b_1}{b_2}$$

Do this for $i = 1, 2, ...$

$$r_2^2 \, {}_2f_3 = r_3^3 \Longrightarrow \; {}_2f_3 = \frac{r_3^3}{r_2^2} = \frac{b_2}{b_3}$$

$${}_nf_{n+1} = \frac{b_n}{b_{n+1}} = \frac{r_{n+1}^{n+1}}{r_n^n}$$

Example: Replicating Portfolios, Spots, and Forwards
Coupon bond:
$M = 5yrs$, $c = 8\%$, $F = \$1,000$.

$$P = ?$$

Have the following \$ 10 PDB prices

$$b_1 = 9.346$$

$$b_2 = 8.654$$

$$b_3 = 7.939$$

$$b_4 = 7.217$$

$$b_5 = 6.561$$

Spot Rates

$$
\begin{aligned}
r_1 &= \left(\tfrac{10}{9.346}\right) &&= 1.07 \\
r_2 &= \left(\tfrac{10}{8.654}\right)^{\frac{1}{2}} &&= 1.075 \\
r_3 &= \left(\tfrac{10}{7.939}\right)^{\frac{1}{3}} &&= 1.08 \\
r_4 &= \left(\tfrac{10}{7.217}\right)^{\frac{1}{4}} &&= 1.085 \\
r_5 &= \left(\tfrac{10}{6.561}\right)^{\frac{1}{5}} &&= 1.088
\end{aligned}
$$

Forward Rates

$$ {}_1f_2 = \frac{(r_2)^2}{r_1} \qquad \text{Also,} \quad {}_1f_2 = \frac{b_1}{b_2} $$

$$ {}_1f_2 = \frac{(r_2)^2}{r_1} = \frac{(1.075)^2}{1.07} = 1.08 \qquad {}_1f_2 = \frac{b_1}{b_2} = \frac{9.346}{8.654} = 1.08 $$

$$ {}_2f_3 = \frac{(r_3)^3}{(r_2)^2} = \frac{(1.08)^3}{(1.075)^2} = 1.09 \qquad {}_2f_3 = \frac{b_2}{b_3} = \frac{8.654}{7.939} = 1.09 $$

$$ {}_3f_4 = \frac{(r_4)^4}{(r_3)^3} = \frac{(1.085)^4}{(1.08)^3} = 1.10 \qquad {}_3f_4 = \frac{b_3}{b_4} = \frac{7.939}{7.217} = 1.10 $$

$$ {}_4f_5 = \frac{(r_5)^5}{(r_4)^4} = \frac{(1.088)^5}{(1.085)^4} = 1.10 \qquad {}_4f_5 = \frac{b_4}{b_5} = \frac{7.217}{6.561} = 1.10 $$

Restriction on FWD Rates

Spots and forwards are just interest rates. Can they be < 1?

Example

$$ b_1 = 0.81, \;\; b_2 = 0.9 $$

$$ {}_1f_2 = \frac{b_1}{b_2} = \frac{0.81}{0.9} < 1 $$

To take advantage:

t	0	1	2
(1) buy PDB_1	-0.81	1	0
(2) sell PDB_2	0.90	0	-1
(3) matress from 1–2	0	-1	1
Net	0.09	0	0

Arbitrage!

Punchline

We must have ${}_nf_{n+1} \geq 1$ or an arbitrage exists

$$b_1 \geq b_2 \geq \ldots$$

Riskless Bond Prices — Arbitrage-free?

Basic Idea: must be PDB prices that are arbitrage free, and work,

$$0 < b_i \leq 1, \;\; b_{i+1} \leq b_i$$

$$P_i = cf_1 b_1 + cf_2 b_2 + \ldots + cf_n b_n$$

Example:

Bond #	maturity	FV	Coupon	Price
1	1	1	0% p.a.	.9524
2	2	1000	10% p.a.	1073.33
3	2	5000	5% p.a.	4921.92

No arbitrage:

$$\{b_1, b_2\} \Longrightarrow 1 \geq b_1 \geq b_2 > 0$$

$$P = \sum_{i=1}^{N} cf_i b_i$$

- To fit bond 1, $b_1 = .9524$.
- To fit bond 2,

$$1073.33 = 100b_1 + 1100b_2 = 100(.9524) + 1100b_2 \Rightarrow b_2 = .8892.$$

- To fit bond 3

$$4921.92 = 250b_1 + 5250b_2 = 250(.9524) + 5250(.8892) = 4906.4$$

ARBITRAGE

STRATEGY?

Purchase the combination of bonds 1 and 2 that replicate bond 3's payoffs and short sell bond 3:

Short 1 bond 3	4921.92	-250	-5250
Long 4.773 bond 2	-4.773(1073.33) =-5123.00	477.30	5250.00
Short 227.3 bond 1	227.3(.9524) =216.48	-227.3	0
Arbitrage Profits:	15.4 (rounding)	0	0 (rounding)

General Idea

Solving for Term–Structure (spot rates/forward rates)

- No arbitrage $\Rightarrow$ consistent PDB prices
  - For all bonds

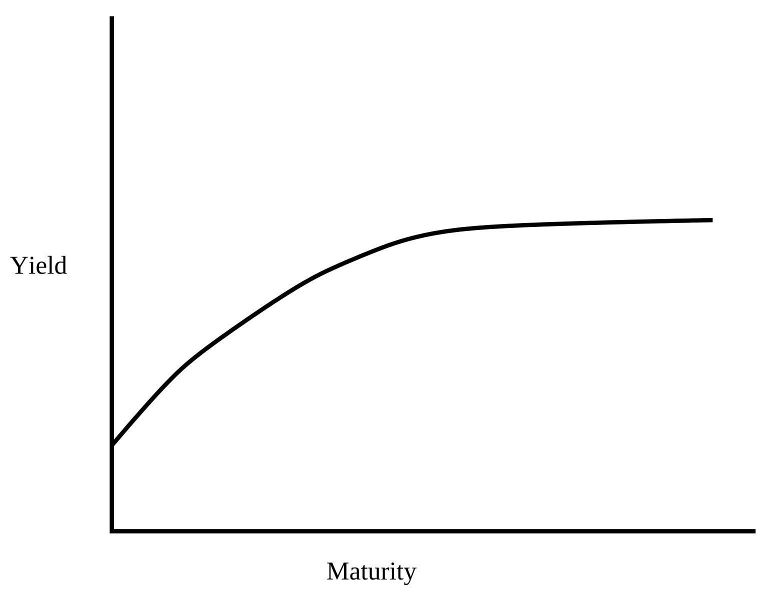
$$P = \sum_i cf_i b_i$$

  - $b_i = 1/(r_i)^i, \; 0 < b_i \leq 1$
  - $b_i \geq b_{i+1}$

$$P_i = cf_1 b_1 + cf_2 b_2 + ... + cf_T b_T$$

- T unknowns
- $\rightarrow$ need T independent equations to get unique solutions
- $< T$ independent equations $\rightarrow$ can't solve uniquely for all b_i
- $> T$ independent equations $\rightarrow$ overidentified

Example of Term Structure



Spot and Forward Rates

Spot rates are average of forward rates:

$$r_t^t = r_{t-1}^{t-1}\ {}_{t-1}f_t = r_{t-2}^{t-2}\ {}_{t-2}f_{t-1}\ {}_{t-1}f_t$$

$$r_t^t = \prod_{j=1}^{t} {}_{j-1}f_j = {}_0 f_1\ {}_1f_2 \cdots {}_{t-1}f_t$$

$$\Rightarrow\ r_t = \left(\prod_{j=1}^{t} {}_{j-1}f_j \right)^{\left(\frac{1}{t}\right)}$$

1. TS increasing ${}_{t-1}f_t > r_t\ \forall t$
2. TS decreasing ${}_{t-1}f_t < r_t\ \forall t$
3. TS flat ${}_{t-1}f_t = r_t\ \forall t$

Yields

Yield: IRR of the bond, or interest rate that equates price and discounted cash flow.

1. P bond price
2. C coupon
3. F face
4. T time to maturity

$$P = \frac{C}{Y} + \frac{C}{Y^2} + \ldots + \frac{F + C}{Y^T}$$

Yield is *DERIVED* from prices!

- $Y = c$ then $P = F$ trading at **PAR**
- $Y > c$ then $P < F$ trading at **discount**
- $Y < c$ then $P > F$ trading at **premium**

Problems with Yields

- Basic tool in bond manager's toolbox
- Should you buy high yield bonds and sell low yield bonds?
- These are riskless flows — does risk matter?
- Yield doesn't say much, unless term structure flat
- Spots vs. Yields

- **NPV**: discount cash flows at the same time at the same rate, and cash flows at different times differently.
  Yield: discounts cash flows differently at the same time, the same at different times.

$$P_A = 10b_1 + 110b_2 = \frac{10}{Y_A} + \frac{110}{Y_A^2}$$

$$P_B = 6b_1 + 6b_2 + 106b_3 = \frac{6}{Y_B} + \frac{6}{Y_B^2} + \frac{106}{Y_B^3}$$

- Two bonds with same maturity and different coupons with different yields:
  $r_1 = 1.1$, $r_2^2 = 1.32$, $r_2 = 1.148$
  Bond A 2yr, 10% coupon, 100 F, Bond B 2yr, 5%, 100 F

$$P_A = \frac{10}{1.1} + \frac{110}{1.32} = 92.42$$

$$P_B = \frac{5}{1.1} + \frac{105}{1.32} = 84.0909$$

$$Y_A : 92.42 = \frac{10}{Y_A} + \frac{110}{Y_A^2} \Longrightarrow Y_A = 1.1464$$

$$Y_B : 84.0909 = \frac{5}{Y_B} + \frac{105}{Y_B^2} \Longrightarrow Y_B = 1.1476$$

NOT THE SAME

- **Yield** assumes you can reinvest at Y

  $r_1 = 1.1$, $r_2^2 = 1.32$, $r_2 = 1.148$

  ${}_1f_2 = \frac{1.32}{1.1} = 1.2$

  Bond A 2yr, 10% coupon, 100 F

$$
\begin{aligned}
P_A &= \frac{10}{1.1} + \frac{110}{1.32} \\
&= \frac{10}{1.1} + \frac{110}{(1.1)(1.2)} \\
&= 92.42
\end{aligned}
$$

Summary

- Spot Interest Rates
- Forward Interest Rates
- Bond Arbitrage
- Yields

Next Time

- Yields
- Measures of Bond Risk
- References:
  - Kritzman on Duration and convexity
  - Textbook: Fixed Income Portfolio Management: Sections 16.1, 16.2