Mean Variance Analysis

Implementation Issues and
Equilibrium

Objectives

- Enough information to implement in Excel
- Practical Issues
- Why factor models are useful for this
- Combining many risky assets with riskless
- CAPM

Multiple Risky Assets

- Vector notation
- ω: N by 1 vector of asset weights
- Σ: variance-covariance matrix of asset returns, N by N matrix
- μ: N by 1 vector of expected returns
- ι: N by 1 vector of ones

Portfolios of Risky Assets

•Portfolio Expected Return:

$$E\left[\tilde{r}_p\right] = \omega'\mu$$

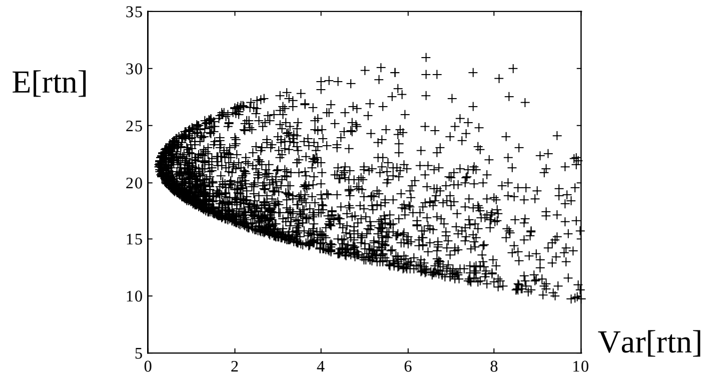
•Portfolio Variance:

$$\sigma_p^2 = \omega'\Sigma\omega$$

•Constraint

$$1 = \omega'\iota$$

Feasible Portfolios



Frontier of Risky Assets

- Minimize portfolio variance subject to
  - mean constraint
    - changing constraint traces out entire curve
  - portfolio constraint

$$
\begin{aligned}
&Min_{\omega}\sigma_m^2 = \omega' \Sigma \omega \\
&st. \\
&\omega' \mu = m \\
&\omega' \iota = 1
\end{aligned}
$$

Properties of Solution

- Can be done in Excel using solver
- First Order Conditions:

$$\Sigma\omega^{*} = 0.5\lambda_1\mu + 0.5\lambda_2\iota$$
$$\omega^{*}`\mu = m$$
$$\omega^{*}`\iota = 1$$

Two Fund Separation

- Useful to compute results
- **All** solutions to the problem are a combination of two portfolios
  - minimum variance portfolio
  - arbitrary optimal portfolio

Minimum Variance Portfolio

- Lowest possible portfolio variance
- Ignore mean constraint
- Use solver in Excel

$$Min_{\omega} \sigma^2_{mv} = \omega' \Sigma \omega$$
$$s.t.$$
$$\omega' \iota = 1$$

How to generate frontier

- Solve for global minimum variance portfolio: r_{mv}
  - compute the mean, m_{mv}
- Pick arbitrary mean, and solve for optimal portfolio at that mean: r_m
- Any other mean, mn
  - weight w, st. w m_{mv}+(1-w)m=mn

Example

- 3 assets
- global minimum variance portfolio
  - [.2,.5,.3]
  - mean at this point 4%
- efficient portfolio with mean of 10%
  - [.5,.1,.4]

Want optimal portfolio w/ mean of 7%?

Variance Calculations

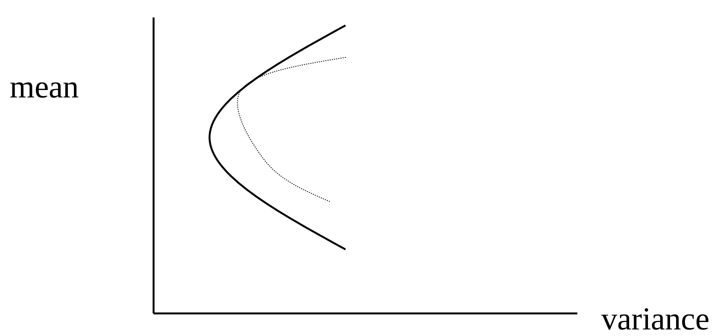
Useful result

- Covariance global minimum variance portfolio and any other portfolio=variance of global minimum variance

Additional Constraints

- Often, fund can't
  - short sell
  - over-invest in sectors
  - use all assets (Green Funds)
- Solution: constraints on problem
- Two-fund result goes away
- Curve moves inside unconstrained frontier

Constrained Frontier



Practical Issues

- You need to get inputs
  - mean vector
  - variance-covariance matrix
- Historical Data?
  - Sampling errors can cause problems

Sampling Error

- Sample average to estimate means and variance covariance matrix

$$\hat{m} = \frac{1}{T}\sum_{t=1}^{T}\tilde{r}_t$$

$$\hat{s}^2 = \frac{1}{T}\sum_{t=1}^{T}(\tilde{r}_t - \hat{m})^2$$

•Estimates have noise!

Example

- Monthly iid returns, stocks
  - 1% per month on average return
  - standard deviation: 4.33 per month
- 2 years of data to estimate returns
  - standard deviation of estimate

$$\sigma(\hat{m}) = \frac{\sigma_m}{\sqrt{T}}$$
$$= \frac{4.33}{\sqrt{24}} = 0.884$$

So What

- 1 standard deviation confidence band
  - 1% ± 0.884%
- Optimal portfolio weights are very sensitive to mean estimates…
- Standard deviation is much more precisely estimated with same amount of data

Solutions

- Don’t only rely on statistics
  - fundamental analysis
  - average with priors…
  - use equilibrium restrictions (Black and Litterman)

Estimating covariance matrix

- Lots of parameters in covariance matrix
  - with 10 assets, 45 parameters!
- Don’t want too long data series
  - stationary issues
- Need to invert covariance matrix
- Solution
  - Factor models….

Factor Models and Portfolio Optimization

- Why
  - reduce number of parameters in var-cov matrix to estimate
- Plausible model

Single Factor Model

- For each asset,

$$\tilde{r}_{i,t} = a + \beta_i \tilde{F}_t + \tilde{\varepsilon}_{i,t}$$

- Asset Variance

$$\sigma_i^2 = \beta_i^2 \sigma_F^2 + \sigma_{\varepsilon_i}^2$$

Covariances of Assets

- Calculate the covariances among assets

$$
\begin{aligned}
Cov(\tilde{r}_i, \tilde{r}_j) &= Cov(\beta_i \tilde{F} + \tilde{\varepsilon}_i, \beta_j \tilde{F} + \tilde{\varepsilon}_j) \\
&= \beta_i \beta_j \sigma_F^2
\end{aligned}
$$

- Putting it all together...

Ways to get factors

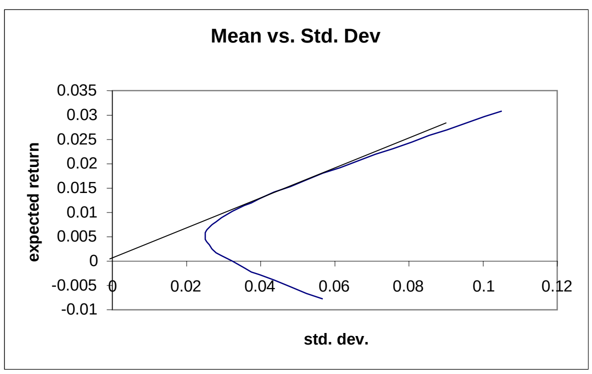
- Factor Analysis
- Economic Factors
  - market portfolio
  - interest rates or term structure
  - inflation
  - oil prices, etc.

Combining with Riskless

- Find efficient portfolio with largest slope

$$\frac{\left(E\left[\tilde{r}_{port}\right]-r_f\right)}{\sigma_{port}}$$

Optimal Portfolios



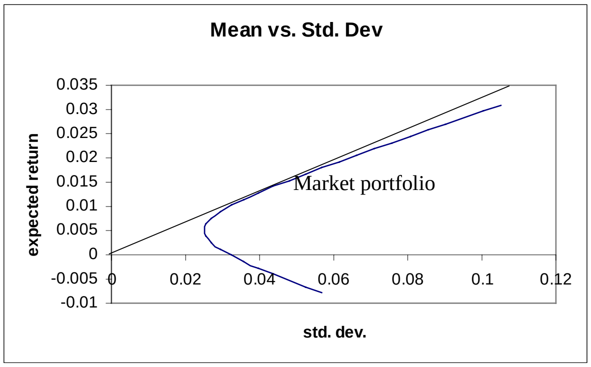
So far...

- Given mean and covariances can find optimal risky position
- Adding riskless
- Practical issues
  - sensitivity
  - where to get inputs

Equilibrium

- Why worry about equilibrium?
  - Benchmark
  - Pricing results
- If we agree on picture
  - all pick same risky position
  - **market portfolio!**

CAPM



So What

- easy to figure out optimal strategy
- Pricing result
- For **all investments, efficient or not calculate beta relative to market**

$$E[\tilde{r}_i] = r_f + \beta_{i,market}\left(E[\tilde{r}_{market}] - r_f\right)$$

Discussion Questions

- You can't get higher return than market if you don't hold an efficient portfolio
- Stocks are riskier than bonds: all stocks should have higher expected return than risk free bonds
- You can't time the market

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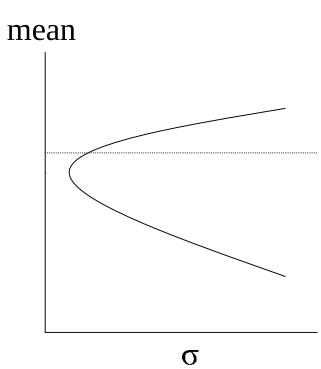
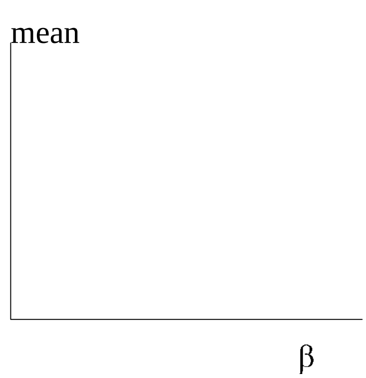
More Questions...





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Yet More

- IBM has a beta of 1
- Is it as risky as market?

Summary

- how to implement
- implementation issues
- factor models again
- equilibrium: CAPM

Next Time

- APT…
- Applications of models: performance evaluation
- Readings:
  - Chapters 11, 24, text
  - Sharpe on Sharpe ratio (readings packet)