Investment Analysis — Homework 1 Solutions

- 1. (a) To replicate one year pdb $\frac{1}{112}$ of coupon bond #1. To replicate the two year pdb, purchase $\frac{1}{112}$ of the two year coupon bond, and short sell $\frac{12}{112}$ of the one year pdb (this is equivalent to short selling $\frac{1}{112} \times \frac{12}{112}$ of the first year coupon bond.
 - (b) The price of the first period pdb = $\frac{1}{1.12} = 0.89$ (since the bond trades at par). The cost of the second period pdb is $\frac{1}{112}97.67 \frac{12}{112} \times \frac{100}{112} = 0.78$.
 - (c) Since the pdb bond prices are positive, and decline as the maturity increases, there is no arbitrage.
 - (d) The bank is offering a forward contract with a rate of 12%. The first period spot rate is 12%, and the forward rate between 1 and 2 is $\frac{b_1}{b_2} = \frac{0.89}{0.78} = 1.14$. Therefore, the bank is willing to lend you money forward at 12%, while you can lend forward at 14%. To take advantage of the bank's offer, borrow forward from the bank and use the coupon bonds to ensure that you can pay the loan off at time 2.

Borrow from bank	0.00	+ 100.00	-112.00
Buy bond $\# 2$	-97.67	+ 12.00	+112.00
Sell bond $\#1$	+100.00	-112.00	0.00
Net	+2.33	0.00	0.00

Thus, the deal is worth 2.33 today. We could also calculate today's value of the deal by taking the present value of the loan from the bank using the current spot interest rates. This will give the same answer.

- 2. (a) Bond 1 has a value of \$5,000 today, since it is trading at par.
 - (b) The yield on bond 1 is $\frac{400}{5000} = 0.08$, or 8%.
 - (c) The equation for bond 2's value is:

$$\frac{160}{(1.082)} + \frac{160}{(1.082)^2} + \frac{1160}{(1.082)^3} = 1200.29$$

(d) To see if there is an arbitrage, we will see if we can find a set of consistent pdb prices. The equations are:

$$5,000 = 400b_1 + 400b_2 + 5400b_3,$$

$$1200.29 = 160b_1 + 160b_2 + 1160b_3$$

To solve these equations, let $x = b_1 + b_2$ and solve for b_3 and x. The equations to be solved are:

$$5,000 = 400x + 5400b_3,$$
$$1200.29 = 160x + 1160b_3.$$

Multiply the first equation by $\frac{4}{10}$ to get

$$2,000 = 160x + 2,160b_3,$$

and subtract the second equation from the result,

$$799.71 = 1,000b_3,$$

so that $b_3 = 0.79971$, and substitute into the first equation to get:

$$5,000 = 400x + 5400(.80)$$

which yields x = 1.70, and so any solutions for b_1, b_2, b_3 must satisfy:

$$b_1 + b_2 = 1.70, \ b_3 = 0.79971$$

and since $1.70 > 2 \times (0.79971)$ we can find a set of pure discount bond prices that are decreasing and price the bonds consistently e.g. $b_1 = .86, b_2 = .84, b_3 = .79971$. Of course, there are many such solutions.

3. (a) The YTM on the first bond solves the equation:

$$0.88 = \frac{1.1}{Y_1} Y_1 = \frac{1.1}{0.88} = 1.25.$$

The YTM on the first bond solves:

$$1.07 = \frac{0.1}{Y_2} + \frac{1.1}{Y_2^2}$$

which yields a value of $Y_2 = 1.0617$.

(b) The system of equations to solve for the PDB bond prices is:

$$\begin{array}{rcl} 0.88 & = & 1.1 \times b_1, \\ 1.07 & = & 0.1 \times b_1 + 1.1 \times b_2. \end{array}$$

The first equation gives $b_1 = \frac{0.88}{1.1} = 0.8$. Substituting into the second equation then gives:

$$1.07 = 0.1 \times 0.8 + 1.1 \times b_2,$$

or $b_2 = 0.9$.

(c) Since the second period PDB bond price is higher than the first period PDB bond price, there must be an arbitrage. This implies that the forward rate between years 1 and 2 is less than one (in gross rates), so to take advantage, we want to borrow at this negative rate. The cash flows we would like to get are given by:

Item	0	1	2
Long 1 period PDB	-0.8	+1	0
Hold between 1 and 2	0	-1	+1
Short 2 period PDB	+0.9	0	-1
Net:	+0.1	0	0

To calculate the required strategy in terms of the coupon bonds, we need to deterine how to replicate PDB give our coupon bonds. We can purchase $\frac{1}{1.1}$ units of bond #1 to replicate a one period PDB. Letting X be the number of units of bond #1 and Y the number of units of bond #2, we need to solve the following system to replicate a two period PDB:

$$X(1.1) + Y(0.1) = 0,$$

$$X(0) + Y(1.1) = 1.$$

From the second equation, $Y = \frac{1}{1.1}$, and substituting into the first equation,

$$X(1.1) + \frac{1}{1.1}(0.1) = 0,$$

which yields $X = -\frac{0.1}{1.1^2} = -0.0826$. Thus, to replicate our strategy of going long 1 period PDB and short 2 period PDB, we need to purchase $\frac{1}{1.1} - \frac{-0.1}{1.1^2} = \frac{1.2}{1.1^2} = 0.9917$ units of bond #1 and purchase $0 - \frac{1}{1.1} = -0.9091$ (or short 0.9091) units of the second period PDB. The cost of this strategy is 0.9917(0.88) + (-0.9091)(1.07) = -0.10, or a positive cash flow of 0.10 today with no future liabilities. You could also solve for this problem by replicated the cash flows of the arbitrage strategy directly with the coupon bonds.

- 4. (a) see #2
 - (b) see #2
 - (c) see #2
 - (d) see #2
 - (e) Today's price of bond #3 is given by $\frac{1000}{1.085} = 921.66$
 - (f) There is no arbitrage with bond #1 and #2 only. However, bond #3 gives us a one period PDB bond price of $\frac{1}{1.085} = 0.9217$. Substituting into the earlier result that $b_1 + b_2 = 1.7$, bond #3 implies that $b_2 = 1.7 0.9217 = 0.7783$. From our earlier results, $b_3 = 0.79971$. Thus, these PDB prices are not consistent, so there is some sort of arbitrage between all three bonds.
 - (g) The three PDB prices calculated earlier, imply that the forward rate is negative between time 2 and 3. To take advantage of this, we want to construct a strategy that has cash flows of zero at time 1, +1 at time 2 and -1 at time 3. Letting X be the number of bond #1, Y bond #2 and Z bond #3, the system of equations to solve is given by:

$$0 = X(400) + Y(160) + Z(1000)$$

+1 = X(400) + Y(160) + Z(0),
-1 = X(5400) + Y(1160) + Z(0).

This system of equations can be solved to yield $X = \frac{-33}{1000}$, $Y = \frac{29}{2000}$ and $Z = \frac{-1}{1000}$. The cost of the strategy is given by

$$\frac{-33}{1000}(5000) + \frac{29}{2000}(1200.29) + \frac{-1}{1000}(921.66) = -0.0175$$

Thus, you receive an arbitrage profit of 0.0175 each time you do this strategy.

- 5. Please see the spreadsheet on the coursepage. I will post this after class.
- 6. (a) Given that the one period PDB is 0.95, the highest possible two period PDB must be 0.95, or the forward rate between 1 and 2 would be negative. So, the lowest possible two-period spot rate is $\frac{1}{0.95}^{\frac{1}{2}} = \sqrt{1.0526315789} = 1.0259783521 = 2.6\%$.
 - (b) To solve this question, you need to realize that a two period PDB must be at least as big as the 4 period PDB of 0.8. Thus, we must have $b_2 \ge b_4 = 0.8$. So, the highest possible one period spot rate of interest is given by $\sqrt{\frac{1}{0.8}} = 1.118$ or 11.80%.

(c) In general, the term structure can only slope down fast enough so that the forward rates are all equal to 1 in gross rates (or 0% in net rates). This is all I wanted for an answer. For those of you who are keen on algebra, the requirement that forward rates never be negative can be written as

$${}_{n}f_{n+1} = \frac{b_{n}}{b_{n+1}} = \frac{r_{n+1}^{n+1}}{r_{n}^{n}} \ge 1,$$

 $r_{n+1}^{n+1} \ge r_n^n,$

which can be rearranged to give

or taking the appropriate roots,

$$r_{n+1} \ge r_n^{\frac{n}{n+1}}.$$

An interesting thing to notice from this equation is that as n gets large, so that we are looking at longer and longer bonds, $\frac{n}{n+1}$ must get close to 1, so the curve must flatten out.