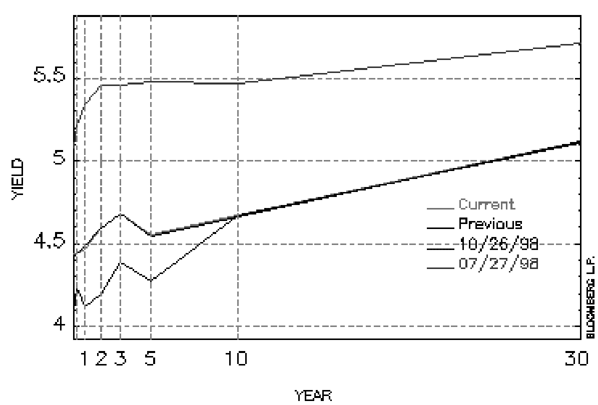
Objectives

- Nonparallel yield curve shifts
  - factor models
  - hedging applications
- Adding Time
- Options
  - what?

Recent TS (Bloomberg)



Main Points

- Non-parallel shifts
- Independence of yield changes across bonds?
- Modify 'hedging' tools to deal with this

Basic Idea

- Make yields 'related' statistically
- Work with continuously compounded yields

$$
\begin{aligned}
P_{bond} &= \frac{C}{\exp(r_1)} + \frac{C}{\exp(2 \times r_2)} + \cdots \frac{C+F}{\exp(\tau \times r_\tau)} \\
&= \frac{C}{\exp(y_{bond})} + \frac{C}{\exp(2 \times y_{bond})} + \cdots \frac{C+F}{\exp(\tau \times y_{bond})}
\end{aligned}
$$

Example

- Reference bond: 5 year zero
- 5 year yield = %10

$$P_{5year} = \frac{1}{\exp(0.1 \times 5)} = .607$$

Duration with continuous compounding

- Works just like before!
- (Slight) adjustment for continuous compounding

$$D = \sum \frac{cf_i}{\exp(y \times i) \times P} \times i$$

Convexity Formula

- Convexity of zero = maturity squared
- example: 5 year PDB, convexity=
- In general, weighted average

$$Conv = \left[\frac{cf_1}{\exp(y)\times P}\right]\times 1 + \left[\frac{cf_2}{\exp(y\times 2)\times P}\right]\times 2^2 + \cdots \left[\frac{cf_\tau}{\exp(y\times \tau)\times P}\right]\times \tau^2$$

One year yield changes

- Change in the 5 year bond yield
  - proportional to change in 1 year bond yield
  - proportionality factor is 1.1
  - 1% change in 1 year

$$\frac{\Delta P_{5year}}{P_{5year}} \approx -5(\Delta r_5)$$
$$= -5\times(1.1\times\Delta r_1)$$
$$= -5\times 1.1\times 1\%$$
$$= -5.5\%$$

How would you hedge now?

- Change in 5 year = 5 times 1.1 times change in 1 year
- 'Adjusted' duration = 5.5
- x% in 5 year and (1-x)% in 1 year

$$\frac{\Delta P_{port}}{P_{port}} \approx x\frac{\Delta P_{5year}}{P_{5year}} + (1-x)\frac{\Delta P_{1year}}{P_{1year}}$$
$$= x \times -D_5 \times 1.1 + (1-x) \times (-1)$$
$$= x \times (-5.5) + (1-x) \times (-1)$$

Example

- Have a liability with a duration of 3
- Hedge with the one year and 5 year PDB
- Using regular duration?
- Using adjusted duration of 5 year = 5.5

$$D_p = xD_5 + (1-x)D_1$$
$$3 = x5.5 + (1-x)1$$
$$\rightarrow x = \frac{2}{4.5}$$

Litterman and Scheinkman

- Extend this idea to multiple sources of movement of yields
- Statistical fitting of sensitivities
- Applied in practice!

What to do in Data?

- Look for 'commonalties' in movements of various zero yields
- Estimate via 'factor' model or regression
- Interpretation

$$r_{i,t} = A_i + \beta_{1,i} F_{1,t} + \cdots \beta_{K,i} F_{K,i,t} + \varepsilon_{i,t}$$

Terminology

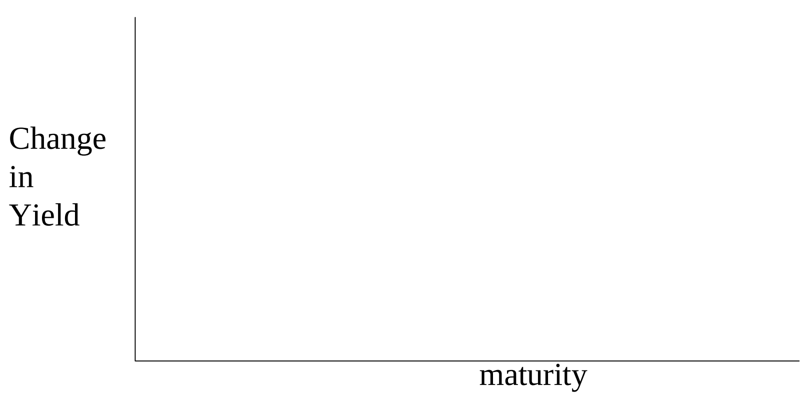
- $F_{k,t}$: factors
  - K of them
  - affect yields systematically
  - zero mean, variance 1, 0 correlation with each other
- $\beta_{k,i}$: factor sensitivity or loading
  - how much does shock to factor k affect yield i

Terminology (cont.)

- $\varepsilon_{i,t}$: residuals, unexplained movements
  - diversifiable
- A_i: Yield with no factor shocks
- Factor Analysis

Estimation Method

Effects of Factor Shocks



Estimates of Model

		Proportion Total Variance Accounted for by:		
Maturity	R-squared	Factor 1	Factor 2	Factor 3
1 year	0.995	79.5	17.2	3.3
2 years	0.982	93.4	2.4	4.2
5 years	0.988	98.2	1.1	0.7
14 years	0.984	86.2	11.5	2.2

Example

- One factor, F
- P=$1 million, D=4 years, PDB
- Use 8 year PDB
- Factor loading: 4 years=1.1, 8 years = 1.2

- Pick weight so that factor adjusted duration = 0

Solution

$$0 = x \times 4 \times 1.1 + (1 - x) \times 8 \times 1.2$$
$$\rightarrow x = \frac{9.6}{5.2}, 1 - x = \frac{-4.4}{5.2}$$

- Go long 229 % in 4 year bond and short 129% in 8 year bond
- What about reverse?

Factor Adjusted Convexity

- What about 'big' factor changes
  - adjust convexity
- Adjusted convexity of 4 year PDB with sensitivity of 1.1?

$$\begin{aligned} C_{adjusted} &= C\left(\beta_1{}^2\right) \\ &= 4^2 \times 1.1^2 \\ &= 19.36 \end{aligned}$$

Portfolio

- Invest x in bond 1 and (1-x) in bond 2
- Portfolio Adjusted Convexity?

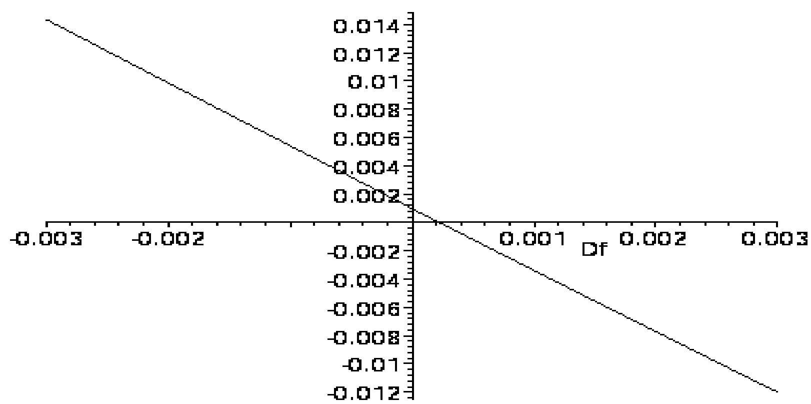
$$C_{port} = xC_1\beta_1^2 + (1-x)C_2\beta_2^2$$

Including Time Value or Putting it all together

- Up to now, ignored time value
- Change in position over small unit of time
- Taylor-series

$$\frac{\Delta P}{P} \approx y\Delta t - D_p\beta_p\Delta F + 0.5C_p\beta_p^2(\Delta F)^2$$
$$= A_p\Delta t - D_p\beta_p\Delta F + 0.5C_p\beta_p^2(\Delta F)^2$$

Return vs. Factor Movements



Portfolio Movements

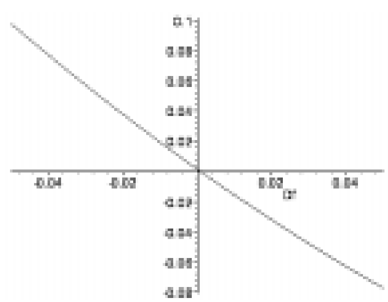
- Invest x in 1 year PDB and (1-x) in 2 year PDB
- 50% in each of the bonds
  - with sensitivities 1.1, and 1.2
- intercepts of 0.001 and 0.002

Computations

$$\frac{\Delta P}{P} \approx (xA_1 + (1-x)A_2) - (xD_1\beta_1 + (1-x)D_2\beta_2)\Delta F$$
$$+0.5(xC_1\beta_1^2 + (1-x)C_2\beta_2^2)(\Delta F)^2$$

$$\approx \big(0.5(0.001) + 0.5(0.002)\big) - \big(0.5(1.1) + 0.5(2)(1.2)\big)\Delta F$$
$$+0.5\big(0.5(1.1)^2 + 0.5(2)^2(1.2)^2\big)(\Delta F)^2$$

Portfolio Returns



Adjusted Convexity vs. Yield

- spread position: short bullet and long barbell
- Modified duration = 0
- Positive convexity
- Intercept?

Introduction to options

- Options are everywhere
  - traded options: basic and exotic
  - interest rate options
  - many securities have embedded options
    - callable debt, convertible debt, warrants, mortgages, etc.
  - Many real investments have option features

Terminology

- Call
- Put
- European style
- American style

Terminology

- Exercising:
- Strike (exercise) price:
- expiration date
- C_t: price of call, P_t: put price, S_t: stock price
- r: risk free rate

Sequence of events for American call

- Now: purchase call for C_0
- Exercise time, T
  - S_T>X: exercise and pay X for stock
  - S_T<X: don't exercise

$$C_T = \begin{cases} S_T - X, \text{ if } S_T > X \\ 0, \text{ otherwise} \end{cases}$$

$$= \max(0, S_T - X)$$

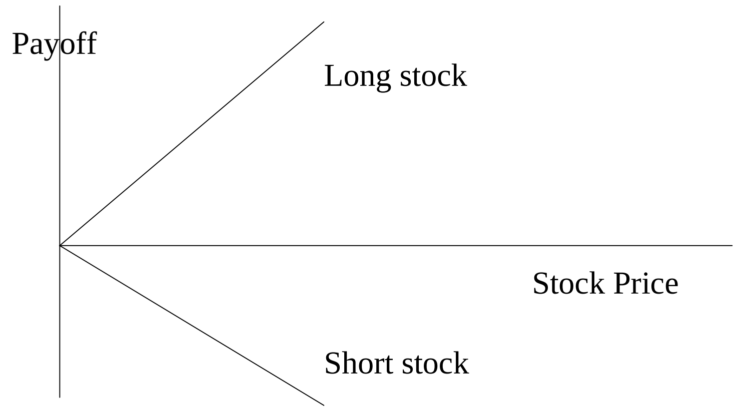
Puts

- Similar timing
- Buying put not the same as writing call

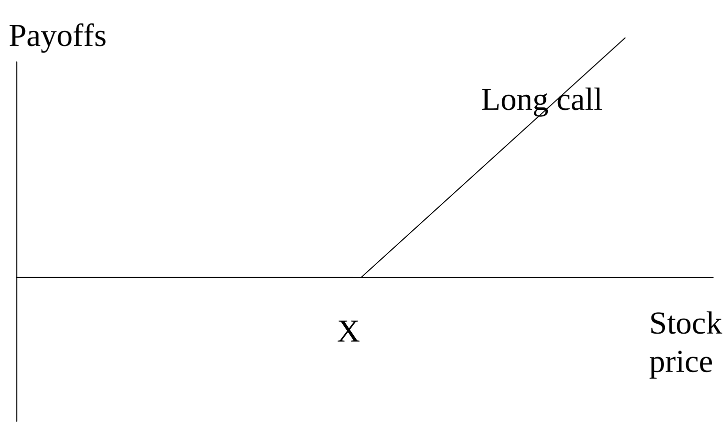
$$P_T = \begin{cases} 0, \quad \text{if} \quad S_T > X \\ X - S_T \quad \text{if} \quad \mathrm{S_T} \le X \end{cases}$$

$$= \max(0, X - S_T)$$

Payoff Diagrams



European call



Summary

- Factor models for bonds
  - what
  - why
  - how to apply
- Adding time
- Introduction to options

Next Time

- Strategies involving options
- Basic option valuation principles
  - Chapter 21, text