CAPM, FACTOR MODELS AND APT

1 CAPM

The CAPM relates the expected return on security i with the excess of the expected return on the market over the risk–free rate, $[Er_m - r_f]$, or the market price of risk. If the CAPM is true, then:

$$Er_i = r_f + [Er_m - r_f]\beta_i \tag{1}$$

where $\beta_i = \frac{cov(r_i, r_m)}{var(r_m)}$.

The β of a portfolio, β_p, is a weighted average of the β's of the securities within the portfolio. So, if there are n securities in a portfolio, and the weight of each security in the portfolio is x_i, then

$$\beta_p = \sum_{i=1}^{n} x_i \beta_i. \tag{2}$$

2 FACTOR MODELS

Suppose that there are a fixed number of sources of risk in the economy. Each source of risk (or randomness) is a *factor* and if there are K factors, each is denoted F_k. Each security is exposed to these sources of risk. The extent of the exposure is known as the *factor loading* and the exposure of security i to factor k is denoted β_{ik}. The incremental expected return per unit of sensitivity for a factor is the same across different stocks. Another way of saying this is that the market price of risk is the same. So, there is a constant price associated with bearing each factor's risk. The return on securities differ because either the factor loadings are different, or there has been a firm–specific idiosyncratic shock. The return on a security i at time t is :

$$r_{it} = A_i + \beta_{i1} F_{1t} + \ldots \beta_{iK} F_{Kt} + u_{it}, \tag{3}$$

where A_i is a constant specific to security i and u_{it} is an idiosyncratic shock, specific to security i. The expected return of this security is just:

$$Er_i = A_i + \beta_{i1} EF_1 + \ldots \beta_{iK} EF_K, \tag{4}$$

notice that $Eu_{it} = 0$, that is, we expect the idiosyncratic shock to be zero (otherwise it would not be a shock), and if the factors are defined so that their expected values are zero, the expected return on the security is just A_i.

If you construct a portfolio of n securities with weights x_i then the return on the portfolio is just:

$$r_{pt} = \sum_{i=1}^{n} x_i r_{it},$$

or

$$r_{pt} = \sum_{i=1}^{n} x_i A_i + \sum_{i=1}^{n} x_i \beta_{i1} F_{1t} + \ldots + \sum_{i=1}^{n} x_i \beta_{iK} F_{it}.$$

Hence, the factor loading of a portfolio on factor k is $\beta_{pk} = \sum_{i=1}^{n} x_i \beta_{ik}$. Further, the return on the portfolio is:

$$r_{pt} = A_p + \beta_{p1} F_{1t} \ldots + \beta_{pK} F_{Kt} + u_{pt}. \tag{5}$$

Many different types of factor models are used. There are three broad classes:

1. Macroeconomic factors, such as inflation, percentage change in industrial production.

2. Statistical factor models which use maximum likelihood on cross–sectional and time series to identify the factors.

3. Fundamental factor models such as book to market, dividend yield.

In *The Three types of Factor Models: A comparison of their explanatory power*, Gregory Connor looks at the explanatory power for these three types of models for US equities. You might find this interesting.

Clearly, the CAPM can be viewed as a single factor model where the only factor is the return on the market.

You can use factor models to

1. Speed up the calculation of a variance/covariance matrix for portfolio composition

2. Risk Management

2.1 Calculating a variance/covariance matrix

If you wish to use a factor approach to construct a variance/covariance matrix to determine the optimal portfolio, then

$$\begin{aligned} cov(r_i, r_j) &= E[r_i - Er_i][r_j - Er_j] && (6)\\ &= E[\beta_{i1}F_{1t} + \ldots \beta_{ik}F_{kt} - \beta_{i1}EF_1 - \ldots - \beta_{ik}EF_k][r_j - Er_j] && (7)\\ &= \textstyle\sum_{k=1}^{K}\sum_{g=1}^{K}\beta_{ik}\beta_{jg}cov(F_{kt}, F_{gt}) && (8) \end{aligned}$$

and

$$var(r_i) = \sum_{k=1}^{K}\sum_{g=1}^{K}\beta_{ik}\beta_{ig}cov(F_{kt}, F_{gt}) + var(u_{it}). \quad (9)$$

There is no covariance of the error terms in the equation for the covariance of the two asset returns. This is because the idiosyncratic shocks are *assumed* to be uncorrelated. It is important to recognize that the only correlation between securities comes about because they all depend on the factors. In particular, the idiosyncratic shocks are uncorrelated. (If they were correlated they would not be shocks but a missing factor). This approach is less computationally intensive than estimating the variance/covariance matrix directly.

2.2 Risk Management

When you are constructing your portfolio, you may wish to ensure that none of your factor exposures are too high, or that they mimic (are exactly equal to) the exposures of a particular index. Suppose that your target factor loadings are $\bar{\beta}_1, \bar{\beta}_2$ (in the two factor case). Then you would add constraints:

$$\begin{aligned} \beta_{1p} &= \bar{\beta}_1 && (10)\\ \beta_{2p} &= \bar{\beta}_2, && (11)\\ & && (12) \end{aligned}$$

to your portfolio optimization problem.

You can also use this framework to isolate different components of returns. Suppose that you felt that the constant term for security i was large. (If you interpret security i as a managed fund then you feel that the manager is doing a good job. After we have discussed evaluation measures you will see why this could be called the alpha of a fund). However, you don't like the factor exposure that comes with security i. One way around this problem is to construct a portfolio with the technique above that has the same factor loadings as the security in which you are interested, call this

portfolio, $\hat{r}_i$. Then, if you form a composite portfolio by buying i and shorting $\hat{i}$, the expected value of what you are left with is:

$$A_i + \beta_{i1}EF_1 + \beta_{i2}EF_2 - \hat{\beta}_{i1}EF_1 - \hat{E}\beta_{i2} = A_i. \tag{13}$$

This technique is called *creating a portable alpha*.

3 ARBITRAGE PRICING THEORY

The APT is a substitute for the CAPM. The CAPM says that there is linear relationship between the expected return on an asset and its covariance with the market portfolio or, that the security market line is straight. The APT says that there can be many factors and for each factor there is a linear relationship between the asset's expected return and it's covariance with the factors. Essentially, the factors represent risk which the investor cannot diversify away and the slope coefficients of the linear relationship between the expected returns and the covariances are interpreted as risk premia. So, the APT says that the expected return on an asset is a linear function of its factor loadings. Or,

$$Er_{it} = \lambda_0 + \lambda_1\beta_{i1} + \lambda_2\beta_{i2} \ldots + \lambda_k\beta_{ik} \tag{14}$$

Compare this with (4).