33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2019, Deserno

Problem sheet #9

To prepare for the midterm on Wednesday, make sure you're up to speed with:

- probabilities and probability densities
- Poisson distribution for discrete random variables, and Gaussian for continuous random variables
- the transformation theorem for probabilities and probability densities
- Jensen's inequality
- Legendre transformations and some of their key properties
- How to make new thermodynamic potentials out of the energy $U(S, V, N)$
- differentials, integrability conditions, Maxwell relations, and integrating factors
- $\mathrm{d}U = T\,\mathrm{d}S - P\,\mathrm{d}V + \mu\,\mathrm{d}N$, and how to get the differentials of other thermodynamic potentials
- partial derivatives and their convenient expression as Jacobians

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31. Adiabatic oscillations *(5 points)*

Imagine a cylinder of cross-sectional area A. It stands on its permanently closed bottom, is filled with some gas (not necessarily an ideal one), and closed at the top by a tightly fitting lid of mass M that can *slide vertically without friction* along the inside walls of the cylinder. In equilibrium, the lid will rest at some height h above the bottom, enclosing a volume $V = Ah$.

1. List all the forces that act on the lid. Which directions do they have? Which of them depend on the lid's vertical position?
2. If you displace the lid away from equilibrium and then release it, it will oscillate with some frequency ω. Assume that the container walls and the lid conduct heat very poorly, hence the periodic compressions and expansions of the gas happen adiabatically. Derive a differential equation for the motion of the lid, valid for small oscillations. What is the frequency?
3. If the cylinder contains a monoatomic ideal gas, show that the frequency can be written as $\omega^2 = \frac{5/3}{Mh}(P_\mathrm{o}A + Mg)$, where P_o is the outside air pressure and g the acceleration of gravity.

   Notice that you could use this device to measure the air pressure P_o, namely, by measuring the oscillation frequency!

32. "A pearl of theoretical physics"... *(5 points)*

...that's what H.A. Lorentz called Boltzmann's following brilliant insight: Consider another mystery system, of which we only know that it is extensive, the chemical potential vanishes, and it satisfies the equation of state $PV = \frac{1}{3}U$.

1. Explain why in such a situation we must have $U(T, V) = V\,u(T)$.
2. Express the entropy as a function of temperature and volume. (This will involve $u(T)$, which you need not eliminate.)

   Hint: The Euler equation will prove useful, but you should explain, why you're allowed to use it!
3. Find a differential equation for $u(T)$ by pondering over the temperature dependence of the pressure. *(Hint: Maxwell!)*
4. Solve the differential equation and thus predict how the energy density and the entropy depend on the temperature.