33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2019, Deserno

Problem sheet #6

19. Closed versus exact differentials—the fine difference *(5 points)*

Consider the following differential form in two dimensions:

$$đf = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2}\,. \tag{1}$$

Show that this differential form is *closed*, but *not exact*. (And why is that?)

In other words: the integrability condition holds, but not all closed loop integrals vanish. This explains what you need to show!

20. The efficiency of real heat engines *(4 points)*

In class we have shown that the efficiency of ideal reversible heat engines is given by the Carnot efficiency $\eta = 1 - T_{\mathrm{C}}/T_{\mathrm{H}}$. But real heat engines must run at finite speeds. They cannot be quasistatic, and hence not completely reversible. Show that this implies that *the efficiency of a real heat engine is less than that of an ideal one*.

Think very carefully about which part in our derivation breaks down, and which one doesn't.

21. Maximum work from a temperature difference *(6 points)*

Suppose we have two buckets of water with constant heat capacities C_{A} and C_{B}, so that the relationship between the change of temperature in bucket i and its change in energy is $\mathrm{d}U_i = C_i\,\mathrm{d}T$. The buckets are initially at temperature $T_{\mathrm{A},0}$ and $T_{\mathrm{B},0}$. We now put an ideal heat engine between these two buckets, depleting that temperature difference to extract mechanical work.

1. What is the final temperature of the water in the two buckets?

   Hint: Start by drawing a diagram of how you connect the buckets and the machine, show your flow of heat and work.

2. What is the maximum amount of work you can extract with such a heat engine from the two buckets?
3. If you just mixed the two buckets of water, instead of using the heat engine, what would be the final water temperature?
4. Is the final temperature in the mixing case higher, lower, or the same as when the heat engine is used? Give *both* a physical argument *and* a mathematical proof of your answer.

   Hint: Your expressions will clear up when you introduce the probability distribution $p_i = C_i/(C_A + C_B)$.

5. Calculate the change in entropy, ΔS, that occurs when the water is simply mixed together, and *prove* that $\Delta S \geq 0$.

22. Another inequality—just for good measure! *(5 points)*

1. Let $p(x)$ and $p_0(x)$ be two continuous probability densities defined on $\mathbb{R}$. Prove Gibbs' inequality

   $$\int \mathrm{d}x\; p(x) \log\big[p_0(x)\big] \;\leq\; \int \mathrm{d}x\; p(x)\log\big[p(x)\big]\,. \tag{2}$$

   Hint: First prove $\log(x) \leq x - 1$. Now bring the right hand side of (2) to the left, combine, and use the log-inequality.

2. The *von Neumann entropy of a probability density* is defined as the following functional:

   $$S[p] \;=\; -\int \mathrm{d}x\; p(x)\log\big[p(x)\big]\,. \tag{3}$$

   Using the Gibbs inequality, prove the following Theorem: *Among all probability densities of the same variance, the Gaussian has the largest von Neumann entropy.*

   Hint: Start with the Gibbs inequality and choose $p_0(x)$ wisely!