33-765 — Statistical Physics

Department of Physics, Carnegie Mellon University, Spring Term 2019, Deserno

Problem sheet #2

5. Poisson distribution *(7 points)*

Another distribution which one frequently encounters is the so-called "Poisson distribution". It is defined by

$$P_\mu(n) = \frac{\mu^n}{n!}\mathrm{e}^{-\mu} \qquad n \in \mathbb{N}_0 \ , \ \mu \in \mathbb{R}^+ \ . \tag{1}$$

Show that $P_\mu(n)$ is properly normalized and calculate its expectation value $\langle n \rangle$ and variance σ_n^2!

Hint: $\langle n^2 \rangle$ is a bit finicky to calculate directly, but $\langle n(n-1) \rangle$ isn't...

6. More on the Poisson distribution *(8 points)*

In problem 5 we encountered the discrete Poisson distribution function $P_\mu(n) = \mu^n \mathrm{e}^{-\mu}/n!$. It is a good model to describe the number n of random events that independently occur in some interval of time, during which the *expected* number if μ.

1. It turns out that one way to think about the Poisson distribution is as follows: consider a Bernoulli process with N trials and success probability p, and imagine the limit in which $N \to \infty$, $p \to 0$, but $Np = \mu = \text{const}$. Show that in this limit the associated binomial distribution function $P_{\text{bin}}(n; N, p)$ converges towards the Poisson distribution $P_\mu(n)$!

2. Check this statement *numerically* by graphically comparing the distribution function $P_{10}(n)$ with several Bernoulli distribution functions of increasingly large N and small p, such that $Np = 10$.

3. And finally: a neat application to the Poisson distribution. A support center receives calls from customers who need help with some product. The calls arrive randomly and independently of each other, but historical data shows that the center receives on average 10 calls per hour. Beyond a certain number of calls in any given hour the support line is overwhelmed and the system collapses, so the center needs to make sure to employ enough operators to handle occasional rushes.

   a) At least how many calls does the support center have to be able to handle within an hour so that the probability of being overwhelmed is less than 0.1%?

   b) Repeat your calculation for three successively bigger call centers that receive on average 20, 50, and 100 calls per hour. Use your findings to argue why large call centers can be run more efficiently than small ones!

   Hint: you will need to calculate these probabilities numerically—there's no (easy) closed expression.

7. Independence versus uncorrelatedness *(5 points)*

1. Prove that if two random variables X and Y are independent, then they are also uncorrelated.

   The definition of independence makes a statement about the joint probability distribution $P_{X,Y}(x,y)$ of the two random variables. Use this to simplify the expression of the covariance between X and Y. Also, remember how the two marginals $P_X(x)$ and $P_Y(y)$ were defined.

2. The reverse is not true. Show by explicit calculation that the probability table given on the right provides a counter-example: two uncorrelated random variables that are *not* independent.

Y \ X	−1	0	+1	
−1	1/8	1/8	1/8	
0	1/8	0	1/8	
+1	1/8	1/8	1/8	