33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Sample Midterm Questions

1. Probability densities and random variables

Let $X \in \mathbb{R}$ be a random variable on the real numbers, whose probability density $p_X(x)$ is given by $p_X(x) = \dfrac{1}{\pi \cosh(x)}$. Without explicitly checking this, you may rest safely assured that $p_X(x)$ is indeed normalized.

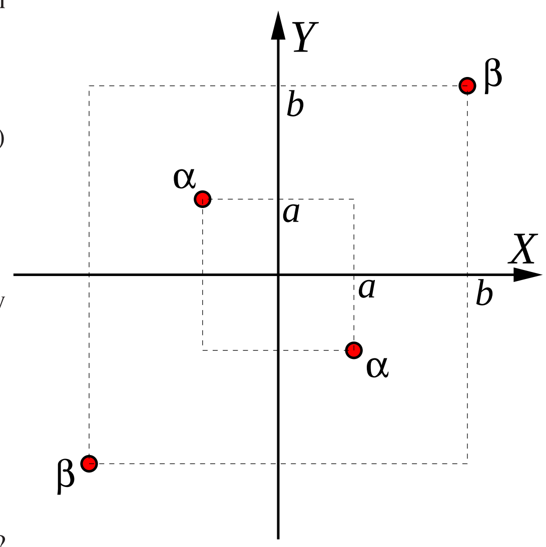
1. Provide a clean sketch of $p_X(x)$. How does it approach zero as $x \to \pm\infty$?
2. Let $\mu_{X,n} := \langle X^n \rangle$ for $n \in \mathbb{N}$ be the n^{th} moment of $p_X(x)$. Without doing an explicit calculation: which of these moments exist, and which of them are nonzero?
3. Let the new random variable $Y := \cosh(X)$ be a function of the old one. Calculate its probability density $p_Y(y)$.
   Hint 1: If we use "arcosh" to denote the inverse function of cosh, believe that $\sinh(\text{arcosh}(y)) = \sqrt{y^2 - 1}$ *for* $y \geq 1$.
   Hint 2: Don't forget to contemplate that functions needn't be uniquely invertible...
4. Provide a clean sketch of $p_Y(y)$. Pay special attention to the domain on which it is defined, asymptotes, and divergences.
5. Let $\mu_{Y,n} := \langle Y^n \rangle$ for $n \in \mathbb{N}$ be the n^{th} moment of $p_Y(y)$. What can you say about these moments?

2. Let the correlations vanish

Consider two random variables X and Y, with a joint probability distribution given as follows:

$$P_{X,Y}(x,y) = \begin{cases} \beta & \text{if } X = b \text{ and } Y = b \\ \alpha & \text{if } X = a \text{ and } Y = -a \\ \alpha & \text{if } X = -a \text{ and } Y = a \\ \beta & \text{if } X = -b \text{ and } Y = -b \end{cases} , \tag{1}$$

where $b > a > 0$. This is also illustrated in the figure on the right.



1. What has to be true for α and β so that this is a proper probability distribution?
2. What are the probability distributions of X and Y?
3. Calculate $\langle X \rangle$ and $\langle Y \rangle$.
4. For which choices of α and β are X and Y uncorrelated?
5. For the case that X and Y are uncorrelated, are they also independent?

3. Under pressure

Let's say that the entropy of some mystery system is given by the expression

$$S(E,V,N) = A\,(EVN)^{1/3} , \tag{2}$$

with some positive constant A. Show that the pressure of the system is equal to its energy density!