33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Problem sheet #13

39. A better thermodynamic model for our atmosphere *(5 points, due on Monday)*

The standard (High School? Physics 101?) model of our atmosphere assumes that the air temperature does not depend on height. This simplified "isothermal atmosphere" assumption for instance gives rise to the very well known barometric height formula $P(h) = P(0)\,\mathrm{e}^{-mgh/k_\mathrm{B}T}$, where $P(h)$ is the atmospheric pressure at a height h above ground, m is the mass of a gas molecule, g is the gravitational acceleration, and T is the supposedly constant temperature. But if you think about this model for a short while, you realize that it is not actually a good approximation at all: it really does get colder when you climb up a mountain! Let's try to do better.

Consider a "packet" of air. If it moves up, it gets into a region of lower pressure, and so it expands. If it expands, it cools, and we will assume that it does this in such a way that overall it doesn't lose any energy to the surrounding air or pick up energy from the surrounding air. This is a pretty good assumption, because the thermal conductivity of air is very poor. The volume changes of the air packet are therefore *adiabatic*, and not isothermal.

We will assume that the air is still in good approximation an ideal gas, but to be a bit more quantitative, we will account for the fact that it is an ideal *diatomic* gas. We will (hopefully) soon learn that the isochoric specific heat for such a gas is $c_V = \frac{5}{2}k_\mathrm{B}$.

1. What is the adiabatic exponent γ for a diatomic ideal gas?
   Hint: recall a very general relation we derived for the difference between c_p and c_V. That will help you to get c_p.

2. Derive a relation between pressure and temperature for an adiabatic process, and from this an equation for $\mathrm{d}P/\mathrm{d}T$.
   Hint: Show that for adiabatic compressions we have $P^{\text{this}}\,T^{\text{that}} = \text{const.}$, and then take the differential of that.

3. The atmospheric pressure at an elevation h above ground results from the weight of that part of the atmosphere that is still above h. *(That smells like an integral…)* Using this fact, derive $P(h)$ and from this $\mathrm{d}P/\mathrm{d}h$.

4. By combining your results from (2) and (3), derive an equation for $\mathrm{d}T/\mathrm{d}h$.

5. Making reasonable assumptions about the molecules that comprise our atmosphere, calculate how much the temperature should decrease if you climb up 100 meters. Does your answer make sense?

40. A better thermodynamic model for our atmosphere — continued *(5 points, due on Tuesday)*

1. In part (4) of the previous problem you derived a differential equation for the temperature. Integrate it and show that the answer can be written as

   $$T(h) = T_0\left(1 - \frac{\gamma-1}{\gamma}\,\frac{h}{h_0}\right) . \tag{1}$$

   What are T_0 and h_0?

2. What are the pressure $P(h)$ and density $n(h)$ as a function of height? Plot these, using suitably rescaled axes!

3. Eqn. (1) shows that in the limit $\gamma \to 1$ the temperature becomes constant. Show that in that case the equations for both pressure and density converge towards the "conventional" barometric height formula of an isothermal atmosphere.
   Hint: you might find it useful to revisit the hint in problem 20!

4. Unlike the isothermal atmosphere, our refined model actually has an "upper boundary" h_max. What is the formula for h_max, and what is its value for our own atmosphere? Does the answer strike you as reasonable?