33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Problem sheet #12

36. A harbinger of insight—the return of an entropy from eons ago *(7 points, due on Monday)*

Let's go back to the mystery system we investigated in problem 18 from homework sheet #6 (or in fact from problem 3 on the first midterm!), but let's introduce alternative parameters and new dimensionless ("reduced", "scaled") variables:

$$V_\mathrm{c} = 3Nb \quad , \quad k_\mathrm{B}T_\mathrm{c} = \frac{8a}{27b} \quad , \quad P_\mathrm{c} = \frac{a}{27b^2} \qquad \text{and hence define} \qquad \tilde{T} = \frac{T}{T_\mathrm{c}} \quad , \quad \tilde{P} = \frac{P}{P_\mathrm{c}} \quad , \quad \tilde{V} = \frac{V}{V_\mathrm{c}} \, . \tag{1}$$

1. Show that in the reduced variables $\tilde{T}$, $\tilde{V}$ and $\tilde{N}$ the thermal equation of state reads $(\tilde{P} + 3\,\tilde{V}^{-2})(3\,\tilde{V} - 1) = 8\,\tilde{T}$.
2. Find the relationship between $\tilde{P}$ and $\tilde{V}$ (with $\tilde{T}$ eliminated!) that holds when the Joule-Thomson coefficient $\mu_\mathrm{JT} = 0$.
(Note: It turns out that for pressures below this so-called "inversion curve" $\tilde{P}_\mathrm{inv}(\tilde{V})$, we have $\mu_\mathrm{JT} > 0$.)
3. Using the scaled thermal equation of state, show that the volume on the inversion curve satisfies $\tilde{V}^{-1} = 3 - \sqrt{4\tilde{T}/3}$.
4. Inserting this into $\tilde{P}_\mathrm{inv}(\tilde{V})$, you get the inversion curve in the $\tilde{T}$-$\tilde{P}$ diagram. Plot it! (The part under the curve is the region that has a positive Joule-Thomson coefficient and will thus cool when subjected to the Joule-Thomson process.)
5. For hydrogen (H_2) we have $T_\mathrm{c} = -240\,^\circ\mathrm{C}$ and $P_\mathrm{c} = 12.7\,\mathrm{atm}$, while for carbon dioxide (CO_2) we have $T_\mathrm{c} = 31.2\,^\circ\mathrm{C}$ and $P_\mathrm{c} = 72.8\,\mathrm{atm}$. Do these gases heat up or cool down under a throttled expansion at room temperature and pressure?

37. "A pearl of theoretical physics"... *(6 points, due on Wednesday)*

...that's what H.A. Lorentz called Boltzmann's following brilliant insight: Consider another mystery system, of which we only know that it is extensive, the chemical potential vanishes, and it satisfies the equation of state $PV = \frac{1}{3}U$.

1. Explain why in such a situation we must have $U(T, V) = V\,u(T)$.
2. Express the entropy as a function of temperature and volume. (This will involve $u(T)$, which you need not eliminate.)
Hint: The Euler equation will prove useful, but you should explain, why you're allowed to use it!
3. Find a differential equation for $u(T)$ by pondering over the temperature dependence of the pressure. *(Hint: Maxwell!)*
4. Solve the differential equation and thus predict how the energy density and the entropy depend on the temperature.

38. The curious thermodynamics of rubber bands *(7 points, due on Friday)*

Stretching an ordinary rubber band of rest length L creates a tension $\tau > 0$ in it (units: Newton). Any subsequent length change $\mathrm{d}L$ does the work $đW = \tau\,\mathrm{d}L$ on the band. The differential of the rubber band's energy is thus $\mathrm{d}U = T\,\mathrm{d}S + \tau\,\mathrm{d}L$.

1. *Experimentally* [sic!] determine the sign of $\left(\frac{\partial T}{\partial L}\right)_S$: Stretch a rubber band quickly (to ensure the change happens close to adiabatically) and use your lips to determine whether it becomes warmer or colder. You can also stretch it, hold it for a while to thermally equilibrate, touch it to your lips, and then rapidly *relax* the extension.
Warning: If the rubber band snaps while you hold it to your lips, it will hurt. Don't blame your instructor, though. (Some sacrifices have to be made for science.)
Hint: If you maxwellify your answer, you will discover the sign of yet another thermodynamic derivative.
2. Now imagine the rubber band being attached to a hook in the ceiling and stretched by hanging a weight at the other end. Let it come to equilibrium. Now heat the rubber band with a hair dryer. Combine your newly acquired thermodynamic superpowers (your knowledge of Maxwell relations, Jacobians, stability conditions, etc.) with what you have learned from your little experiment and *predict, whether the heated rubber band will lift or lower the weight.* Obviously, this question only asks you about a sign; hence, *getting the sign right is everything*!
a) Evidently, we're interested in the equivalent of the thermal expansion coefficient α. How would you define it here?
b) You need to Jacobi-expand the quantity of interest in terms of objects for which you know the sign. The expansion is non-standard, but considering for what quantities you know the sign (recall the stability conditions!) should help.