33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Problem sheet #10

30. A nontrivial (but super instructive) Legendre transform. I. *(5 points, due on Monday)*

Consider the function $f_a(x)$, which contains a parameter $a \in \mathbb{R}$, and its Legendre transform $f_a^\star(p)$:

$$f_a(x) \;=\; -\frac{1}{2}ax^2 + \frac{1}{4}x^4 \qquad , \qquad f_a^\star(p) \;=\; \min_x \big\{ f_a(x) - px \big\} \; . \tag{1}$$

Studying the Legendre transform $f_a^\star(p)$ is nontrivial, because depending on the value of a the function $f_a(x)$ is *not* everywhere convex. Always keep in mind that the value of a might qualitatively change the results, so be careful about this.

1. Find any minima, maxima, and inflection points of $f_a(x)$. For which values of a is the function always convex? Produce a careful sketch of $f_a(x)$ for typical representative cases, in which these interesting points are visible.
2. In order to actually perform the Legendre transform, you need the equation that links p and x. Find it!
3. The *graph* of $f_a^\star(p)$ is the collection of points $\{p, f_a^\star(p)\}$. Neglecting in a first step the "min" prescription in the Legendre transform, consider the collection of points $\{p(x), f_a(x) - p(x)x\}$, which you can view as a *parametric representation* of the graph of $f_a^\star(p)$ (with x being the parameter). Using your favorite plotting program, plot that graph for representative values of a. What happens when you tune a such that $f_a(x)$ ceases to be convex? Which bits of the (possibly funny-looking) graph of $f_a^\star(p)$ will "survive" the application of the "min" in the Legendre transform that we have ignored so far? What therefore happens to the Legendre transform $f_a^\star(p)$ once $f_a(x)$ deviates locally from convexity?

31. A nontrivial (but super instructive) Legendre transform. II. *(5 points, due on Tuesday)*

Let's continue with problem 30:

1. If we want to analytically determine $f_a^\star(p)$, we need to solve the equation linking p and x for x. This, unfortunately, is a bit messy, but the solution can be written down reasonably neatly. First let's define $r^2 = 4a/3$ and $\cos(3\alpha) = 4p/r^3$. Now show (without using MATHEMATICA or relatives!) that the following three values solve the equation:

$$x_0 \;=\; r\,\cos(\alpha) \qquad , \qquad x_1 \;=\; r\,\cos\left(\alpha + \frac{2\pi}{3}\right) \qquad , \qquad x_2 \;=\; r\,\cos\left(\alpha + \frac{4\pi}{3}\right) \; . \tag{2}$$

   Hint: The trigonometric identity $4\cos^3(A) = 3\cos(A) + \cos(3A)$ should come in handy.

2. Identify the three solutions with the three interesting branches of the Legendre transform which show up once $f_a(x)$ is no longer convex. Feel free to use your favorite plotting program to do that; no formal proof is required.
3. What is $f_a^\star(0)$, $\lim_{p\to 0^+} f_a^{\star\prime}(p)$ and $\lim_{p\to 0^-} f_a^{\star\prime}(p)$? *(Hint: recall what's true about a Legendre transform's derivative!)*

32. Maxwell relations and Jacobians in tedious disguise. *(5 points, due on Wednesday)*

1. Prove the first $T\,\mathrm{d}S$ equation: $\;T\,\mathrm{d}S = Nc_V\,\mathrm{d}T + \dfrac{\alpha T}{\kappa_T}\,\mathrm{d}V$.
2. Prove the second $T\,\mathrm{d}S$ equation: $\;T\,\mathrm{d}S = Nc_P\,\mathrm{d}T - \alpha T V\,\mathrm{d}P$.

Hint: If you write the entropy $S(\cdot,\cdot,N)$ in the variables indicated by the $T\,\mathrm{d}S$ equation(s), what would be its differential?

33. Adiabatic compression *(5 points, due on Friday)*

1. Show that, quite generally, $\dfrac{\kappa_T}{\kappa_S} = \dfrac{c_P}{c_V} =: \gamma$, where γ is called the *adiabatic index*.
2. Calculate κ_T, c_V, c_P, and γ for the monoatomic ideal gas.
3. Show that adiabatic (constant entropy) compression of an ideal gas leads to a pressure-volume relation $P \propto V^{-\gamma}$.