33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Problem sheet #9

27. A few relatively simple Legendre transforms *(5 points, due on Monday)*

Reminder: the Legendre transform $f^\star(p)$ of a function $f(x)$ is defined as $\min_x\{f(x) - px\}$, if $f(x)$ is convex, and as $\max_x\{f(x) - px\}$, if $f(x)$ is concave.

1. Calculate the Legendre transform $f^\star(p)$ and its derivative $f^{\star\prime}(p) = \partial f^\star(p)/\partial p$ for the following functions:
   a) $f(x) = \mathrm{e}^x$
   b) $f(x) = \log(x)$
   c) $f(x) = \cosh(x)$
2. Consider the family of functions $f_\alpha(x) = \frac{1}{\alpha}x^\alpha$ with $x \in \mathbb{R}^+$ and $\alpha \in \mathbb{R}\backslash\{0, 1\}$.
   a) For which values of α is $f_\alpha(x)$ convex and for which is it concave?
   b) Calculate the Legendre transform $f^\star_\alpha(p)$ of $f_\alpha(x)$.

28. One slightly less simple Legendre transform *(5 points, due on Tuesday)*

Calculate the Legendre transform $f^\star(p)$ and its derivative $f^{\star\prime}(p) = \partial f^\star(p)/\partial p$ for the function $f(x) = \dfrac{x^2}{1+|x|}$. What are the domains over which $f(x)$ and $f^\star(p)$ are defined? Also: plot both functions!

Ponder: is this even differentiable? Is it convex or concave?

Hint: You might first want to look separately at positive and negative x, before you put Humpty Dumpty back together again!

29. Maximum work from a temperature difference *(10 points, due on Wednesday)*

Suppose we have two buckets of water with constant heat capacities C_A and C_B, so that the relationship between the change of temperature in bucket i and its change in energy is $\mathrm{d}U_i = C_i\,\mathrm{d}T$. The buckets are initially at temperature $T_{\mathrm{A},0}$ and $T_{\mathrm{B},0}$. We now put an ideal heat engine between these two buckets, depleting that temperature difference to extract mechanical work.

1. What is the final temperature of the water in the two buckets?

   Hint: Start by drawing a diagram of how you connect the buckets and the machine, show your flow of heat and work.
2. What is the maximum amount of work you can extract with such a heat engine from the two buckets?
3. If you just mixed the two buckets of water, instead of using the heat engine, what would be the final water temperature?
4. Is the final temperature in the mixing case higher, lower, or the same as when the heat engine is used? Give *both* a physical argument *and* a mathematical proof of your answer.

   Hint: Your expressions will clear up when you introduce the probability distribution $p_i = C_i/(C_A + C_B)$.
5. Calculate the change in entropy, ΔS, that occurs when the water is simply mixed together, and *prove* that $\Delta S \geq 0$.

We'll be busy inaugurating our new president on Friday.

Hence; no classes and no homework!