33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Problem sheet #8

24. Can you handle the truth? *(5 points, due on Monday)*

Consider the following differential in two dimensions:

$$đf = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{x^2 + y^2} \qquad , \quad (x, y) \in \mathbb{R}^2 \backslash \{0\} \,. \tag{1}$$

1. Show that the integrability condition for differentials that we derived in class does in fact hold for $đf$. (The technical lingo for this is: "the differential is *closed*".)

2. If a differential is "*exact*", this means that a function g exists such that $\mathrm{d}g$ is equal to that differential. In order to get g, we have to integrate the differential from some initial point to the final point (x, y). This, however, is only well-defined if the answer does not depend on the path taken—for otherwise that function g would not depend merely on the end point (x, y) but instead on the entire path taken to get there. Show that this condition of path-independence is equivalent to the vanishing of all closed loop line integrals.

3. Now show that your professor's mumblings about exactness requiring more than the integrability conditions being true were not entirely paranoid: *the closed differential (1) is not exact*. To verify this explicitly, calculate the line integral

$$L := \oint đf \,, \tag{2}$$

where the closed-loop integral is taken counterclockwise over a circle of radius R, centered at the origin. What do you have to conclude, based on the outcome of this calculation?

Moral: The condition you checked in part (1) is necessary, *but not* sufficient. *Specifically, it is a local condition, while the vanishing of all closed-loop integrals really is a global statement. To link these two, we need some extra information of topological nature. Here, it suffices if we also know that the space on which the differential form is defined is "simply connected". But alas, that's not true for $đf$. (Why?)*

25. The efficiency of real heat engines *(5 points, due on Tuesday)*

In class we have shown that the efficiency of ideal reversible heat engines is given by the Carnot efficiency $\eta = 1 - T_\mathrm{C}/T_\mathrm{H}$. But real heat engines must run at finite speeds. They cannot be quasistatic, and hence not completely reversible. Show that this implies that *the efficiency of a real heat engine is less than that of an ideal one*.

Think very carefully about which part in our derivation breaks (and which part doesn't).

26. All reversible heat engines have the same efficiency *(5 points, due on Wednesday)*

Let's assume that there exists a *reversible* heat engine M with an efficiency $\eta_\mathrm{M} < \eta_\mathrm{Carnot}$ ("M" could stand for "miracle" or "mediocre", depending on how you prefer to look at it). In this problem you are asked to show that the existence of such a machine would violate the second law of thermodynamics, and hence no such machine can exist.

1. Assume you use the machine with η_M as a heat pump. What would be its coefficient of performance, $\epsilon_\mathrm{M,HP}$?

2. Couple both the machine M and an ordinary Carnot engine between two reservoirs, a hot one at temperature T_H and a cold one at temperature T_C. Connect them so that the Carnot machine runs as a heat engine, whose work output is used to run the M-machine as a heat pump. Show that the overall effect of this combined contraption does something that violates (some of the many versions of) the second law of thermodynamics.

   Hint: Draw a picture that shows the two reservoirs, both machines, and all flows of heat and work. This will help you keep track of minus signs and quickly grasp what is going on.

3. Show that the statement can be generalized: in order to violate the second law, all you need is two reversible heat engines 1 and 2 with unequal efficiencies $\eta_1 \neq \eta_2$ (none of which might have the Carnot efficiency $1 - T_\mathrm{C}/T_\mathrm{H}$). Hence, you have thus proved that *all reversible heat engines have the same efficiency, which therefore is the Carnot efficiency*.