33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Problem sheet #6

18. Equilibrium between two gases *(5 points, due on Monday)*

Imagine a container that is divided by a wall into two compartments. The left compartment contains N particles of an ideal gas, the right compartment also contains N particles, but the system is not an ideal gas. Instead, it has the following entropy:

$$S(E,V,N) \;=\; Nk_{\rm B}\left[\frac{3}{2}\ln\frac{E+aN^2/V}{N}+\ln\frac{V-Nb}{N}+X\right] \,, \tag{1}$$

where $a>0$, $b>0$ and X are constants.
We assume that the wall is impermeable to particles, but that it permits the exchange of energy between the two compartments, and that it can also freely slide, so that the volumes taken by the two compartments can change.

1. What are the equilibrium conditions that need to be satisfied between the two compartments?
2. Show that the mystery system in the right compartment always has an energy that's lower than that of the ideal gas.
3. What is the volume that the ideal gas will occupy? Is the answer unique?

First midterm on Wednesday!
No homework for Tuesday and Wednesday!

19. The Cauchy distribution, revisited *(10 points, due on Friday)*

We will look again at the Cauchy distribution, this time adding a scale parameter a:

$$p_X(x;a) = \frac{a}{\pi}\,\frac{1}{a^2+x^2} \qquad \text{with} \;\; a>0\,. \tag{2}$$

1. Show that $p_X(x;a)$ is correctly normalized.
2. As an intermediate result, show the following: if any random variable Z has the characteristic function $\tilde{p}_Z(k)$, then the random variable nZ has characteristic function $\tilde{p}_Z(nk)$.
3. Believe the following result:
$$\int_{-\infty}^{\infty}{\rm d}x\,\frac{{\rm e}^{{\rm i}kx}}{1+x^2} \;=\; \pi\,{\rm e}^{-|k|} \qquad \text{for} \;\; k\in\mathbb{R}\,. \tag{3}$$
Use this to calculate the characteristic function $\tilde{p}_X(k;a)$ of the Cauchy distribution $p_X(x;a)$!
4. Let X_1 and X_2 be two independent random variables, both distributed according to Eqn. (2). What is the characteristic function of the distribution belonging to X_1+X_2?
5. Let $\{X_1,X_2,\ldots,X_n\}$ be n independent random variables, all distributed according to Eqn. (2). What is the characteristic function belonging to the distribution of $X_1+X_2+\cdots+X_n$?
6. Let $\overline{X}_n=\frac{1}{n}(X_1+X_2+\cdots+X_n)$ be the arithmetic mean of n independent random variables, all distributed according to Eqn. (2). What is the distribution function of $\overline{X}_n$?