33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Problem sheet #4

11. More on the Poisson distribution *(6 points, due on Monday)*

In problem 9 we encountered the discrete Poisson distribution function $P_\mu(n) = \mu^n \mathrm{e}^{-\mu}/n!$. It is a good model to describe the number n of random events that independently occur in some interval of time, during which the *expected* number if μ.

1. It turns out that one way to think about the Poisson distribution is as follows: consider a Bernoulli process with N trials and success probability p, and imagine the limit in which $N \to \infty$, $p \to 0$, but $Np = \mu = \text{const}$. Show that in this limit the associated binomial distribution function $P_{\text{bin}}(n; N, p)$ converges towards the Poisson distribution $P_\mu(n)$!

2. Check this statement *numerically* by graphically comparing the distribution function $P_{10}(n)$ with several Bernoulli distribution functions of increasingly large N and small p, such that $Np = 10$.

3. And finally: a neat application to the Poisson distribution. A support center receives calls from customers who need help with some product. The calls arrive randomly and independently of each other, but historical data shows that the center receives on average 10 calls per hour. Beyond a certain number of calls in any given hour the support line is overwhelmed and the system collapses, so the center needs to make sure to employ enough operators to handle occasional rushes.
   a) At least how many calls does the support center have to be able to handle within an hour so that the probability of being overwhelmed is less than 0.1%?
   b) Repeat your calculation for three successively bigger call centers that receive on average 20, 50, and 100 calls per hour. Use your findings to argue why large call centers can be run more efficiently than small ones!

   Hint: you will need to calculate these probabilities numerically—there's no (easy) closed expression.

12. Histograms for functions of random variables *(5 points, due on Tuesday)*

Download the program `histogram-of-RV.py`. It creates a histogram for a random variable X that is distributed according to a Gaussian with expectation value μ and standard deviation σ, and it also plots the expected number of counts in the bins.

1. Run the program and show an example output. Explain the meaning of the variable `normalization`!

2. Adjust the program so that it instead plots a histogram for the **sum** $Z = X + Y$ of two Gaussian random variables X and Y. Also find the new function that again describes the histogram. Show the output.

3. Same as 12.2, but for the **ratio** $Z = X/Y$ of two Gaussian random variables X and Y, in the special case where both X and Y have expectation value 0 and standard deviation 1. Is it a Gaussian? Show the output.

13. Jensen's inequality *(4 points, due on Wednesday)*

Let $f : \mathbb{R} \supset G \to \mathbb{R}$, $x \mapsto f(x)$ be a function which is *convex* (some say: "convex up") and (for simplicity) differentiable on the compact connected domain G. As a consequence, for all $x, x_0 \in G$ we have $f(x) \ge f(x_0) + f'(x_0)\,(x - x_0)$.

1. Draw an educationally pristine graph to illustrate that this inequality makes sense as a definition of convexity.

2. Let X be a continuous random variable and $f(x)$ be a convex (differentiable) function. Prove *Jensen's inequality*:

$$\big\langle f(X) \big\rangle \;\ge\; f\big(\langle X \rangle\big) \;, \tag{1}$$

   where the average is taken over whatever p-density underlies the randomness of X.
   Hint: Begin by choosing a suitable x_0 in the convexity inequality.
   Additional note: this inequality is often <u>extraordinarily</u> useful, because it is so general. It is well worth remembering!

14. Ratio of two Gaussians *(5 points, due on Friday)*

Let X and Y be two Gaussian random variables with expectation value 0 and standard deviation 1. Define the random variable $Z = X/Y$ and calculate its probability density $p_Z(z)$. Adjust the program you developed in problem 12.3 so that you can plot the histogram of Z together with your prediction for it, in order to check that you got the answer right! Show the output.
Hint: this obviously calls for the transformation theorem, and some ideas for how to solve the resulting integral.