33-341 — Thermal Physics I

Department of Physics, Carnegie Mellon University, Fall Term 2018, Deserno

Problem sheet #3

7. The Gamma-function *(5 points, due on Monday)*

One of the first "special functions" one learns about in college—after having covered exponentials, logs, and trigonometric functions in high school—is probably the Gamma function, $\Gamma(x)$, which is usually defined as follows:

$$\Gamma(x) = \int_0^\infty \mathrm{d}t \; t^{x-1}\mathrm{e}^{-t} \qquad x \in \mathbb{R},\; x > 0 \qquad \text{(in fact: } x \in \mathbb{C},\; \mathrm{Re}(x) > 0) \,. \tag{1}$$

1. Prove the recurrence relation $\Gamma(x+1) = x\,\Gamma(x)$. *Hint: integration by parts...*
2. Prove that if N is a non-negative integer, $\Gamma(N+1) = N!$. *Hint: induction...*
3. Calculate the value of $\Gamma(\frac{1}{2})$. *Hint: clever substitution...*

8. Saddle-point approximation for the Gamma function *(5 points, due on Tuesday)*

If $f(t)$ is a sufficiently smooth function with a unique maximum at t_m, we can Taylor-expand it around that point to get

$$f(t) = f(t_\mathrm{m}) - \frac{1}{2}\big|f''(t_\mathrm{m})\big|(t-t_\mathrm{m})^2 + \mathcal{O}(t^3) \,. \tag{2}$$

1. Write the integrand in the definition of the Gamma function in the form $\mathrm{e}^{f_x(t)}$, where x is just an additional parameter. For which values of x will $f_x(t)$ have a unique maximum? Find that maximum, and Taylor expand up to quadratic order. This turns the integrand into a Gaussian function. Now extend the range of integration from $-\infty$ to $+\infty$ and evaluate the integral. What you get is called the "saddle point approximation" for the Gamma function. What is it?
2. What should hold so that extending the range of integration all the way to $-\infty$ is not a bad approximation?
3. What approximation for $N!$ do you get, when you use the saddle-point-approximation of the Gamma function?

9. Poisson distribution *(4 points, due on Wednesday)*

Another distribution which one frequently encounters is the so-called "Poisson distribution". It is defined by

$$P_\mu(n) = \frac{\mu^n}{n!}\mathrm{e}^{-\mu} \qquad n \in \mathbb{N}_0 \,,\; \mu \in \mathbb{R}^+ \,. \tag{3}$$

Show that $P_\mu(n)$ is properly normalized and calculate its expectation value $\langle n\rangle$ and variance σ_n^2!
Hint: $\langle n^2\rangle$ is a bit finicky to calculate directly, but $\langle n(n-1)\rangle$ isn't...

10. Gaussian approximation of the binomial distribution *(6 points, due on Friday)*

Consider the binomial distribution $P(n,N;p)$ with success probability $p = \frac{1}{2}$ and an even number of trials N.

1. Show that $\langle n\rangle = \frac{1}{2}N$ and $\sigma_n = \frac{1}{2}\sqrt{N}$.
2. Calculate $\log[P(\langle n\rangle, N, \frac{1}{2})]$ using the Gaussian approximation of the binomial distribution.
3. Calculate $\log[P(\langle n\rangle, N, \frac{1}{2})]$ for large (even) N by using Stirling's approximation from Eqn. (3.72) in the textbook.
4. Improve your calculation from part (3) by using Gosper's version of Stirling's approximation—Eqn. (3.73) in the textbook. What is now the difference between your improved result for $\log[P(\langle n\rangle, N, \frac{1}{2})]$ and the Gaussian approximation from part (2)? Give the lowest order approximate expression for that difference, valid for large values of N.
   Hint: $\log(1+x) = x + \mathcal{O}(x^2)$ is useful in the end...