Hyperbolic Orbits for $1/r^2$ forces

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The two branches of the hyperbola shown in the accompanying figure are solutions to the equation

$$x^2/a^2 - y^2/b^2 = 1, \tag{1}$$

and can be represented parametrically as

$$x = \pm a \cosh \tau, \quad y = b \sinh \tau \tag{2}$$

for $-\infty < \tau < \infty$; $-$ refers to the curve on the left, $+$ to the one on the right.

The alternative polar representation employs the focus at $x = -\epsilon a$ and $y = 0$, and takes the form

$$r = \frac{\alpha}{\epsilon \cos \theta\ + 1}, \quad \hat{r} = \frac{\alpha}{\epsilon \cos \hat{\theta}\ - 1} \tag{3}$$

for the branches on the left and right, respectively, with

$$\epsilon^2 = 1 + \left(\frac{b}{a}\right)^2 = 1 + \frac{2El^2}{\mu k^2}. \tag{4}$$

Also note that

$$a = \frac{\alpha}{\epsilon^2 - 1}, \quad b = \frac{\alpha}{\sqrt{\epsilon^2 - 1}}, \tag{5}$$

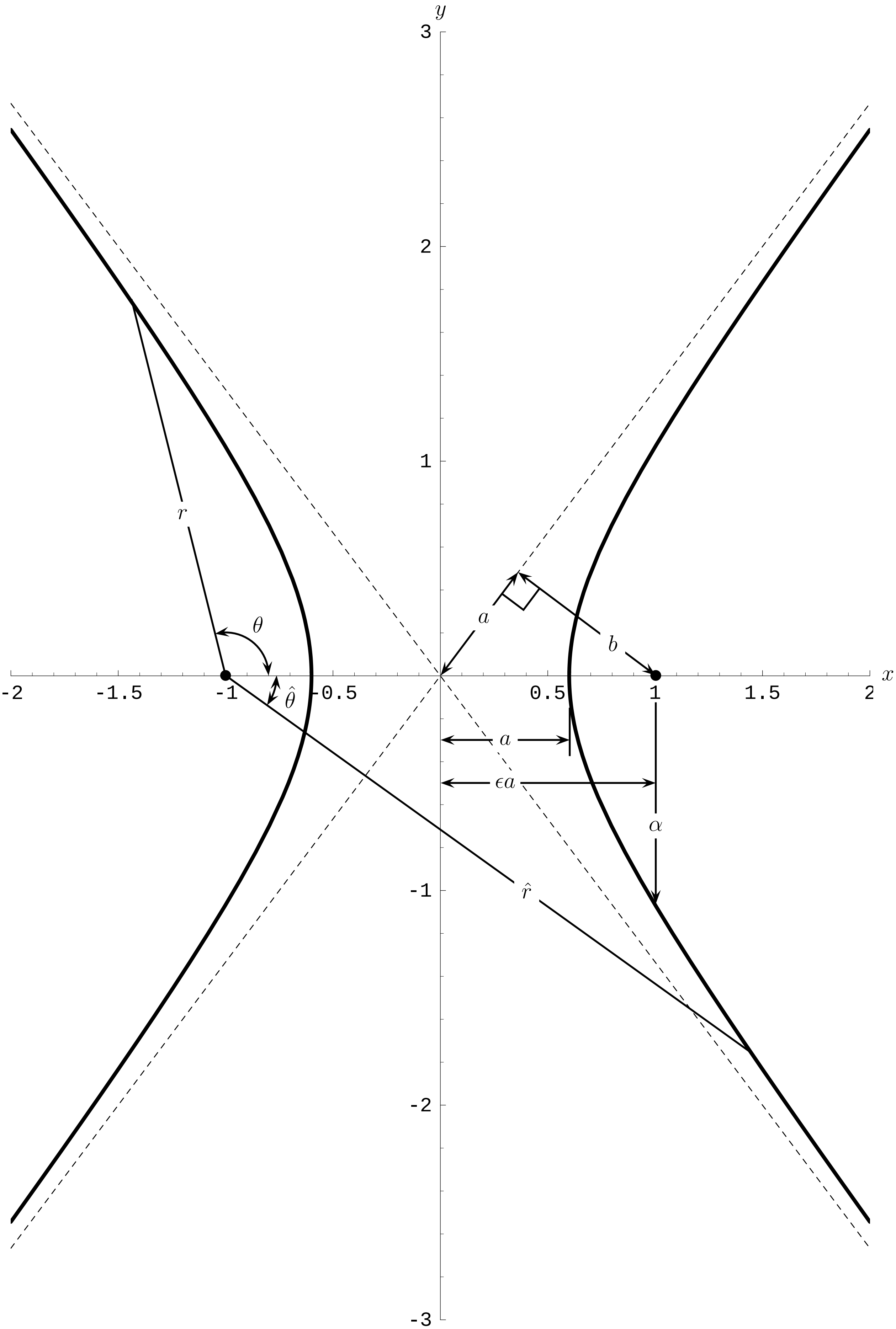
and

$$E = \frac{k}{2\alpha}(\epsilon^2 - 1) = \frac{k}{2a}, \tag{6}$$

where $k > 0$ for either the attractive $(-k/r^2)$ or repulsive $(+k/r^2)$ case.

If the attractive center is at the left focus it is the left branch of the hyperbola that is of physical relevance for an attractive potential (gravity or opposite electric charges), whereas the right branch is relevant for a repulsive potential (charges of the same sign).

The asymptotes are indicated by dashed lines, and each focus is a perpendicular distance b from both asymptotes. In scattering theory b is called the *impact parameter*. The slopes of the asymptotes are $\pm b/a$



y
x
3
2
1
-1
-2
-3
-2
-1.5
-1
-0.5
0.5
1
1.5
2
r
θ
θ̂
r̂
a
b
a
εa
α