Tidal Forces

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Reference:
TM = Thornton and Marion, *Classical Dynamics*, 5th edition

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1 Principle of Equivalence

★ Principle of equivalence: The *inertial* mass m in $\mathbf{F} = m\mathbf{a}$ is the same as the *gravitational* mass m in $\mathbf{F} = m\mathbf{g}$ or $F = -GmM/R^2$. This is a highly nontrivial observation which has been the subject of careful experimental tests, and is fundamental to the general theory of relativity.

★ Consider astronauts in a closed box at rest in a uniform gravitational field $\mathbf{g}$. There is absolutely no way (without going outside the box) they can distinguish this situation from one in which the box is in zero gravitational field but is being pushed by a rocket in such a way that it is accelerated with acceleration $\mathbf{g}$.

★ Similarly, if a closed box is falling freely, thus accelerating, in a uniform gravitational field, someone inside the box cannot distinguish this situation from one in which the box is at rest in empty space.

• One way to express this is in terms of the gravitational potential $\Phi(\mathbf{r})$. In a uniform gravitational field of strength g_u we have

$$\Phi(x, y, z) = g_u z, \tag{1}$$

where the z coordinate is in the opposite direction to the gravitational acceleration. However, in an *accelerated* coordinate system with acceleration g_u in the negative z direction, the gravitational potential is $\Phi' = 0$ (or $\Phi' = C$ for some constant C). Thus a gravitational potential of the form (1) can be removed by shifting to a suitable accelerated coordinate system in which a freely-falling body is at rest.

2 Nonuniform Gravity

★ Suppose the closed box containing the astronauts is at rest on the surface of a spherically symmetrical asteroid of mass M_a and radius R_a that is not rotating. If the surface gravitational acceleration of the asteroid is g, can clever astronauts distinguish *this* situation from being pushed by a rocket with uniform acceleration g?

• Yes they can. A pendulum suspended near the top of the box will have a slightly different period from one suspended near the bottom. A plumb line on the left side will not be exactly parallel to one on the right.

• These small deviations from what occurs in a perfectly uniform gravitational field are said to be due to *tidal forces*, i.e., this is what one means by the term "tidal force." Since tidal forces are really gravitational forces, they are proportional to m and are, in effect, accelerations.

★ Next imagine a small cloud of dust, of such low density that the gravitational attraction of the dust particles for each other can be ignored, that is initially at rest some distance above the surface of the asteroid, and starts to fall freely towards its center. Each dust particle will be accelerated with the gravitational acceleration $\mathbf{g}$ that corresponds to its location, independent of what is happening to the other particles.

• If we use an accelerated coordinate system $\mathcal{C}$ in which a reference particle in the middle of the cloud is at rest, then in $\mathcal{C}$ the other dust particles will appear to be accelerated by small amounts which can be ascribed to tidal forces.

• The tidal force on the reference dust particle is by definition zero, since we are using a coordinate system in which this particle is at rest. A dust particle below the reference particle (i.e., closer to the asteroid) will behave (in the coordinate system $\mathcal{C}$) as if it is being accelerated downwards, one above the reference particle will behave as if it is being accelerated upwards, while particles on either side of the reference particle, at the same height above the asteroid, will behave as if they are being accelerated towards the reference particle.

□ Exercise. Verify all of these statements. Can you make them quantitative in terms of how the tidal forces or accelerations vary with distance from the reference dust particle?

3 Quantitative Tidal Forces

★ Set up a Cartesian coordinate system x, y, z with origin at the center of the astronaut's box resting on the surface of the asteroid, with z pointing away from the center of the asteroid.

Expand $\Phi(x, y, z)$, the gravitational potential due to the asteroid, as a power series for small values of x, y, and z:

$$\Phi(x, y, z) = C + gz + b\big[(x^2 + y^2)/2 - z^2\big], \tag{2}$$

where we are ignoring third and higher order terms, such as x^3 or xz^2. Recall that the gravitational force on a mass m is $\mathbf{F} = -m\nabla\Phi$ i.e., the gravitational acceleration is $\mathbf{g} = -\nabla\Phi$. Here C is a constant whose value is of no particular interest, so we could set it equal to zero.

∘ The linear gz term in (2) tells us that at the center of the box the gravitational acceleration is g in a direction downwards towards the center of the asteroid. This is why there are no terms linear in x or y, i.e., z is the vertical direction.

• The form of the quadratic terms in (2) is determined by two considerations. The first is symmetry: Φ is invariant under rotation about the z axis. This means that the xy, xz, and yz terms must be absent, and the coefficient of x^2 must be the same as that of y^2. So the quadratic contribution to Φ must be of the form $a(x^2 + y^2) - bz^2$ with suitable choices of a and b, which could be positive or negative.

□ Exercise. Explain how symmetry gets rid of the xy, xz, and yz terms in (2).

• In addition, $\Phi(x, y, z)$ satisfies $\nabla^2\Phi = 0$ everywhere outside the asteroid, since Φ is the gravitational potential of the asteroid, and outside the asteroid its mass density is zero. This means that $a = b/2$.

□ Exercise. Show it.

∘ To be sure, there will be a contribution to the total gravitational potential from the box and its contents, but we are ignoring this. Astronauts making extremely precise measurements might have to take it into account.

• The quadratic terms in (2) are often said to represent a *quadrupole* force or potential, because of the way this part of Φ varies with angle on the surface of a sphere of radius $r = \sqrt{x^2 + y^2 + z^2}$. It involves an $l = 2$ spherical harmonic, in contrast to the *dipole* term gz, which involves an $l = 1$ spherical harmonic.

★ At a distance R from the center of the asteroid its gravitational potential is $\Phi = -GM/R$, provided $R > R_a$, the radius of the asteroid. From this it follows that

$$g = GM/R^2, \quad b = GM/R^3, \tag{3}$$

when the origin of coordinates in (2) is at some $R > R_a$.

∘ Note that b has dimensions of acceleration/length, or inverse time squared.

□ Exercise. Derive the expressions in (3)

★ One can also use (2) for the gravitational potential inside the dust cloud above the asteroid surface introduced in Sec. 2, where the origin of coordinates is the reference particle at the center of the cloud.

• However, if one uses the accelerated coordinate system $\mathcal{C}$ in which the reference dust particle is at rest it is necessary to replace Φ in (2) with a potential Φ' in which the gz term is set equal to zero, while the other ("tidal") terms remain the same.

• One can then check that the tidal forces, or accelerations, given by $-\nabla\Phi'$ have the same general character discussed at the end Sec. 2. In particular, because b is positive, the forces in the $z = 0$ plane are "attractive" and dust particles are accelerated towards the reference particle, whereas along the z axis they are "repulsive."

□ Exercise. Compare the tidal accelerations given by $-\nabla\Phi'$ with those obtained by simply assuming each dust particle is being accelerated towards the center of the asteroid, independent of all the other dust particles. (See the exercise at the end of Sec. 2.)

4 Tidal Effects of the Moon on the Earth

★ We will discuss the tidal effect of the moon at the surface of the earth by writing the total gravitational potential as

$$\Phi = \Phi_m + \Phi_e, \tag{4}$$

where Φ_m is the contribution from the matter making up the the moon, and Φ_e from that making up the earth.

• Choose a coordinate system x, y, z whose origin is the center of the earth, with the negative z axis passing through the center of the moon. Then we can expand Φ_m for values of x, y, and z small compared to the distance D between the earth and the moon in the form (2), since in the vicinity of the earth $\nabla^2\Phi_m = 0$, with g set equal to g_m, the acceleration at the position of the earth produced by the moon's gravity.

• However, in choosing the origin of coordinates at the center of the earth, we are actually using an accelerated coordinate system, since the center of the earth is accelerating towards the moon with an acceleration g_m. Therefore, we should use a gravitational potential

$$\Phi'_m(x, y, z) = b(x^2 + y^2)/2 - bz^2, \tag{5}$$

where

$$b = GM_m/D^3 = 8.66 \times 10^{-14} \text{ s}^{-2}. \tag{6}$$

○ For more on this accelerated coordinate system, see Sec. 5 below.

□ Exercise. Check the numerical value of b using appropriate values for M_m and D. (They appear in an equation on p. 203 of TM.)

★ The earth's gravitational potential is given by $\Phi_e = -GM_e/r$, as long as $r = \sqrt{x^2 + y^2 + z^2}$ is greater than the radius R_e of the earth. Adding this to Φ'_m gives the total gravitational potential

$$\Phi' = b(x^2 + y^2)/2 - bz^2 - g_e R_e^2/r, \tag{7}$$

where we have replaced GM_e with $g_e R_e^2/r$, where $g_e = 9.8$ m s^{-2} is the gravitational acceleration at the earth's surface. The prime on Φ' reminds us we are using an accelerated coordinate system.

★ Imagine that the solid earth is a perfect sphere of radius R_e, that it is not rotating, and is entirely covered by a thin ocean. What will be the shape of the top of the ocean in the presence of tidal forces from the moon? It will be an equipotential surface $\Phi' =$ constant, since otherwise water would flow down the potential gradient.

• Let us assume that the distance of the ocean surface from the center of the earth is

$$r = R_e + h, \tag{8}$$

where $h \ll R_e$ depends on the geographical location. By symmetry, given our idealized earth, h can only depend upon θ, the angle from the $+z$ axis in polar coordinates, with $\theta = 180°$ the direction to the moon.

• Inserting (8) in (7), using the fact that h/R_e is small, and setting Φ' equal to a constant we arrive at

$$h(\theta) = h_0 + (bR_e^2/g_e)(\cos^2\theta - \tfrac{1}{2}\sin^2\theta), \tag{9}$$

where the numerical value of bR_e^2/g_e is 0.36 m.

☐ Exercise. Fill in the details in the derivation of (9).

☐ Exercise. Using (6), check the numerical value of bR_e^2/g_e.

• What (9) tells us is that the surface of the ocean on our nonrotating earth is 0.36 m higher than average at both $\theta = 180°$ and $\theta = 0$, which is to say right under the moon and on the opposite side of the earth from the moon, and 0.18 m lower at the intermediate $\theta = 90°$ points, with a total variation in height of 0.54 m.

★ But of course the real earth rotates, and it carries the real oceans along with it. So what does the above have to do with the real tides? Good question. The basic point is that the tidal forces due to the moon, which produce the bulging of the ocean in our idealized model, will be pulling and pushing the real oceans as the earth rotates under the real moon, so that pressure gradients will be produced which drive the water in the oceans with a period of about half a day, as seen from a rotating earth, which is what is observed. Our calculation explains this period through the fact that the tidal forces produced by the moon have a quadrupole character: the ocean is not only pulled upwards on the side of the earth directly under the moon, as one might have expected, but also pushed upwards on the opposite side of the earth.

5 Accelerated Coordinate System

★ One might worry that the transformation from Φ_m to Φ'_m in (5) could be invalid because the direction of the acceleration due to the moon's gravity is changing in time, since the moon is moving. It is therefore helpful to construct the accelerated coordinate system explicitly and check what is going on. For simplicity we ignore the sun, and assume the earth and moon are rotating around their common center of mass.

∘ The notation differs from TM Sec. 5.5, but the basic idea is the same.

• Let $\mathbf{R} = (X, Y, Z)$ be an inertial coordinate system. It is convenient, though not necessary, to assume its origin is the center of mass of the earth-moon system. It is not rotating, i.e., the stars are always in the same direction.

• Let $\mathbf{r} = (x, y, z)$ be a coordinate system whose origin is at the center of the earth, but whose orientation is the same as the $\mathbf{R}$ system, thus the two are related by

$$\mathbf{r} = \mathbf{R} - \mathbf{R}_e(t): \quad x = X - X_e(t), \quad y = Y - Y_e(t), \quad z = Z - Z_e(t), \tag{10}$$

where $\mathbf{R}_e = (X_e, Y_e, Z_e)$ is the position of the center of the earth in the $\mathbf{R}$ coordinate system. It follows that velocities and accelerations in the $\mathbf{r}$ and $\mathbf{R}$ systems are related by

$$\mathbf{v} = \mathbf{V} - \mathbf{V}_e(t), \quad \mathbf{a} = \mathbf{A} - \mathbf{A}_e(t), \tag{11}$$

in an obvious notation.

★ The total gravitational potential $\Phi(\mathbf{R}, t)$ as a function of the inertial coordinates $\mathbf{R}$ will be a function of time because both the earth and the moon, which are the sources of the gravitational potential, are moving. The gravitational acceleration at a given time is given by

$$\mathbf{G} = -\nabla\Phi. \tag{12}$$

○ We are using a nonrelativistic approximation in which light and gravity travel with infinite speed. This is adequate for our calculations, though not for NASA's.

• Now define the modified gravitational potential

$$\Phi'(\mathbf{r}, t) = \Phi'(x, y, z, t) = \Phi(\mathbf{r} + \mathbf{R}_e, t) + \mathbf{A}_e \cdot \mathbf{r}. \tag{13}$$

Note that $\mathbf{A}_e$, the acceleration of the center of the earth, is a function of time but not of position.

• It follows from (13) that

$$\mathbf{G}' = -\nabla\Phi' = \mathbf{G} - \mathbf{A}_e, \tag{14}$$

that is, if we use Φ' in place of Φ, we will get exactly the same acceleration, except for subtracting off that of the center of the earth.

○ A slightly subtle point. The ∇ in (14) is $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$, whereas in (12) it is $(\partial/\partial X, \partial/\partial Y, \partial/\partial Z)$. But it follows from (10), along with the fact that the partial derivatives are taken with time fixed, that $\partial/\partial x = \partial/\partial X$, etc., so it is safe to use the same symbol ∇ in both places.

★ Thus the fact that the moon is moving does not alter our analysis in Sec. 4 when we go from Φ to Φ'. However, the moon's motion means the tidal forces will change with time in a nonrotating coordinate system in which the direction of the fixed stars is always the same.

• In particular, our choosing the negative z axis to pass through the center of the moon, while convenient for doing the mathematics in Sec. 4, could give rise to confusion, since it suggests we might be using a rotating coordinate system, with all the complications arising from the corresponding fictitious forces or accelerations; see TM Ch. 10. Instead, think of the x, y, z system as having a fixed orientation, and wait for the exact instant in time at which the moon is along the $-z$ axis. At that moment the tidal forces computed in Sec. 4 are demonstrably correct.

6 The Roche Limit

★ If a satellite held together by its own gravity gets too close to a massive gravitating body (e.g., Jupiter) it can be pulled apart by tidal forces. The general idea can be understood in the following way. Let the massive body have radius R and mass M, and assume the satellite is a sphere of radius r and mass m. If the satellite is at a distance D from the center of the massive body it will be subject to tidal accelerations, of which the most severe occur at the point on its surface closest to the massive body, and the one that is furthest away, with magnitude, see (5) and (6), $2br = 2GMr/D^3$ and a direction opposite to the satellite's own gravitational acceleration at its surface, which is Gm/r^2. Equating the two magnitudes gives

$$m/r^3 = 2M/D^3. \tag{15}$$

Under this condition a rock at the corresponding point on the surface of the satellite is no longer held down by the satellite's gravity, and if thrown upwards will keep moving.

• Let the average density of material in the massive body and in the satellite be $\rho_M = (3/4\pi)M/R^3$ and $\rho_m = (3/4\pi)m/r^3$, respectively. Then we can rewrite (15) as

$$D/R = 2^{1/3}(\rho_M/\rho_m)^{1/3}, \tag{16}$$

and interpret it to mean that a satellite that gets closer than D to the massive body will tend to be broken up by the tidal forces. This limiting distance is known as the *Roche limit*, named for the French mathematician Edouard Roche who studied it in 1848.

◦ Note that the Roche limit applies only to satellites held together primarily by gravitational forces. The satellites launched by NASA are held together by chemistry, not gravity, and tidal forces are no worry.

• The factor of $2^{1/3}$ in (16) depends on our assumption that the satellite is a sphere. But one would hardly expect a satellite that is about to break apart from tidal forces to retain a spherical shape. Thus the actual limit depends on more than the average density of material. If the satellite consists of an incompressible fluid, so is easily squeezed out of shape, the experts tell us that one should replace $2^{1/3} = 1.26$ in (16) with 2.44.

◦ Tidal forces will tend to distort a squishy satellite into a prolate spheroid with its major axis in the direction pointing towards the massive body. But making the satellite longer in this direction will render the tidal forces that are trying to pull it apart even more effective, since they increase with distance from the center of the satellite. Thus we can understand qualitatively why 2.44 is greater than 1.26.