INSTRUCTIONS. This examination consists of two problems. The first, with 3 parts, is worth 60 points, and the second, with 2 parts, is worth 40 points. It is important to give a brief indication of your reasoning. Answer in a way which shows that you understand the subject. If you do not understand what a question means, or think additional formulas would be helpful, please ask the instructor.

TR-0330 Hamiltonian, phase space for vertical motion in gravity

F08 exm3.1 60pts

1. (3 parts, total of 60 points) A particle of mass $m$ moves vertically in a uniform gravitational field. Its potential energy is $U = mgy$, $y$ is the height measured vertically upwards, and $p$ is the vertical momentum, positive when the particle is moving upwards.

   a) Write down the Hamiltonian—you do not need to derive it, though you may if you wish—as a function of $y$ and $p$ (treat this as a one-dimensional problem). Then employ Hamilton’s equations to derive a differential equation for the acceleration $\ddot{y}$. Show your work.

   b) Find a formula for $y$ as a function of $p$ along a trajectory in the $y,p$ phase plane. [Hint: What is $dy/dp$ along such a trajectory? Or you can find the formula if you know both $y(t)$ and $p(t)$.]

   c) Sketch two such trajectories in the $y,p$ plane, with $y$ horizontal and $p$ vertical, using both positive and negative values of $p$. Indicate by an arrow or arrows on each trajectory the direction the phase point moves as time increases. Now consider the region $R$ in the phase plane bounded by your two trajectories and two horizontal lines, $p = p_1$ and $p = p_2$, with $0 < p_1 < p_2$. What happens to this region as time increases from $t = 0$ to some later time $t = \tau$? (I.e., as each point in the initial region, thought of as representing the possible state of a system at $t = 0$, moves according to the equations of motion to a later time $t = \tau$.) Relate this to Liouville’s theorem. [Hint: Something very simple happens as time progresses to a line in the phase plane whose initial position is defined by $p =$constant.]

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2. (2 parts, total of 40 points) For a particle moving in an attractive central potential $U(r)$ the effective potential

$$V(r) = U(r) + \frac{l^2}{2\mu r^2}$$

for a particular angular momentum $l = \mu r^2 \dot{\theta}$ as shown in the sketch. The particle moves in a bound orbit, not necessarily an ellipse, with $r_1$ the minimum and $r_2$ the maximum distance of approach to the center. In polar coordinates the kinetic energy is $T = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2)$.

a) Indicate how by using the sketch of $V(r)$ you can obtain $r_1$ given the value of $r_2$ and $V(r)$. Explain what you are doing. Are any conservation laws involved?

b) Let $\lambda < 1$ be the ratio $r_1/r_2$, let $\dot{\theta}_1$ and $\dot{\theta}_2$ be the angular velocities when $r = r_1$ and $r_2$, and $T_1$ and $T_2$ the corresponding kinetic energies. Find the ratios $\dot{\theta}_1/\dot{\theta}_2$ and $T_1/T_2$ in terms of $\lambda$ and possibly other quantities. For the $T_1/T_2$ ratio start from the fact that in polar coordinates $T = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\theta}^2)$. Indicate your reasoning.