Two masses $m_1, m_2$ on frictionless table

Lagrangian $L = T - U$ is given by

$$T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) - \frac{1}{2} k \left[ \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} - r_0 \right]^2$$

Solve $\vec{r} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$, $\vec{r}_1 = \vec{r}_2 - \vec{r}_1$, $\vec{R} = (x, y)$, $\vec{r} = (r, \theta)$

for

$$\vec{r}_1 = \vec{R} - \frac{m_2}{m_1 + m_2} \vec{r}, \quad \vec{r}_2 = \vec{R} + \frac{m_1}{m_1 + m_2} \vec{r}$$

$$\dot{r}_1 = \dot{R} - \frac{m_2}{m_1 + m_2} \dot{r}, \quad \dot{r}_2 = \dot{R} + \frac{m_1}{m_1 + m_2} \dot{r}$$

$$\frac{1}{2} m_1 (\dot{r}_1^2 + \dot{\theta}_1^2) + \frac{1}{2} m_2 (\dot{r}_2^2 + \dot{\theta}_2^2) = \frac{1}{2} (m_1 + m_2) \dot{\vec{R}} \cdot \dot{\vec{R}}$$

$$+ \frac{1}{2} \left[ \frac{m_1 m_2}{(m_1 + m_2)^2} + \frac{m_1 m_2}{(m_1 + m_2)^2} \right] \dot{r}_1 \dot{r}_2 + \left[ \frac{-m_1 m_2}{m_1 + m_2} + \frac{m_1 m_2}{m_1 + m_2} \right] \dot{\theta}_1 \dot{\theta}_2$$

Therefore

$$T = \frac{1}{2} (m_1 + m_2) V^2 + \frac{1}{2} \mu (r^2 \dot{\theta}^2 + \dot{r}^2)$$

when $V^2 = \dot{x}^2 + \dot{y}^2 = \dot{R}^2$, $\dot{r}^2 = \dot{r}_1^2 + \dot{r}_2^2 = (r^2 \dot{\theta}^2 + \dot{r}^2)$, $\mu = \frac{m_1 m_2}{m_1 + m_2}$

$$L = T - U = \frac{1}{2} M (\dot{x}^2 + \dot{y}^2) + \frac{1}{2} \mu (r^2 \dot{\theta}^2 + \dot{r}^2) - \frac{1}{2} k (r - r_0)^2$$
b) Using \( p_j = \frac{\partial L}{\partial q_j} \), we identify four momenta:

\[
\begin{align*}
    p_x &= \frac{\partial L}{\partial x} = M x, \\
    p_y &= \frac{\partial L}{\partial y} = M y, \\
    p_r &= \frac{\partial L}{\partial r} = m r \dot{r}, \\
    p_\theta &= \frac{\partial L}{\partial \theta} = m r^2 \dot{\theta}.
\end{align*}
\]

Since \( L \) is independent of \( x, y, \) and \( \theta \), therefore \( p_x, p_y, p_r \) are conserved quantities. The components of linear momentum, and the angular momentum about the center of mass. That they are constant follows from Lagrange's equations:

\[
\frac{\partial L}{\partial q_j} = \frac{d}{dt} \frac{\partial L}{\partial q_j} = \frac{d}{dt} p_j.
\]

\( \frac{\partial L}{\partial q_j} = 0 \Rightarrow p_j \) is constant.

In addition, since \( \frac{\partial L}{\partial t} = 0 \), the Hamiltonian

\[
H = \sum_j p_j \dot{q}_j - L = T + U \]

is a constant of the motion.
c) The condition for circular orbits is that

\[ V_{\text{eff}}(r) = \kappa(r) + \frac{l^2}{2ur^2} = \frac{1}{2} k(r-r_0)^2 + \frac{l^2}{2ur^2} \]

should have a minimum for some choice of \( l \).

Differential:

\[ V_{\text{eff}}'(r) = k(r-r_0) - \frac{l^2}{ur^3} = 0 \]

or \[ k(r-r_0) r^3 = \frac{l^2}{u} \]. As \( l^2 \geq 0 \) there are no solutions for \( r < r_0 \) (\( r \) must be nonnegative). For \( r = r_0 \) there is a solution only if \( l = 0 \), but then the masses are stationary in the center of mass: no circular orbit. For any \( r > r_0 \) there will be some circular orbit, and since

\[ V_{\text{eff}}'' = k + \frac{3l^2}{ur^4} > 0 \], it is stable [is a minimum]

For \( m_1 / m_2 = \frac{1}{2} \), \( \vec{r} = 0 \) we have from (a)

\[ \vec{r}_1 = -\frac{2}{3} \vec{r}, \quad \vec{r}_2 = \frac{1}{3} \vec{r} \]

\[ \dot{\vec{r}}_1(t) = -\frac{2}{3} \dot{\vec{r}}(t), \quad \dot{\vec{r}}_2(t) = \frac{1}{3} \dot{\vec{r}}(t) \], so if \( \vec{r}(t) \) traces out a circle the situation is as shown in the sketch at the right, where at a particular time \( t \), \( \vec{r}_1(t), \vec{r}_2(t) \) are as indicated by the dots.
a) In order to escape from

Thanthus the kinetic energy of

the spacecraft must be increased

even so that the total energy E

is 0 (parabola orbit) or \( \frac{1}{2} m v_0^2 \)

if it is desired to achieve a hyperbolic orbit

with speed \( v_0 \) far from Thanthus. The best way of increasing

the kinetic energy is to make \( \Delta v \) in the same direction as

the instantaneous velocity on the circular orbit, as per the

sketch, because the change in kinetic energy

\[
\Delta T = \frac{1}{2} m (v_0 + \Delta v)^2 - \frac{1}{2} m v_0^2 = m (v_0 \Delta v - \frac{1}{2} \Delta v^2)
\]

is in the greatest possible.

The initial energy in a circular orbit in a \( 1/r \)

gravitational potential is

\[ E_0 = T_0 + U_0 = -T_0 \]

so to achieve a parabolic orbit one needs \( \Delta T = T_0 \),

or \( \frac{1}{2} m (v_0 + \Delta v)^2 = m v_0^2 \) or \( v_0 + \Delta v = \sqrt{2} v_0 \),

or \( \Delta v = (\sqrt{2} - 1) v_0 = 0.414 v_0 \). But to achieve a hyperbolic

orbit with speed \( v_0 \) far away, we need \( \Delta T = 2T_0 \),

which means \( \frac{1}{2} m (v_0 + \Delta v)^2 = \frac{3}{2} m v_0^2 \)

\( v_0 + \Delta v = \sqrt{3} v_0 \) \( \Delta v = (\sqrt{3} - 1) v_0 = 0.732 v_0 \).
b) If the rocket is fired in a direction perpendicular to the one in (a), the new kinetic energy will be given by

\[ T = \frac{1}{2} m \left( \vec{V}_0 + \vec{\Delta V} \right)^2 \]

\[ = \frac{1}{2} m V_0^2 + \frac{1}{2} m (\Delta V)^2 \]

and \( \Delta T = \frac{1}{2} m (\Delta V)^2 \)

Hence to achieve a parabolic orbit, \( \Delta T = T_0 \), one will need \( \Delta V = V_0 \), and to achieve a hyperbolic orbit with speed \( V_0 \) a great distance from Thantus will require \( \Delta T = 2T_0 \), and thus \( \Delta V = \sqrt{2} V_0 = 1.414 V_0 \).
c) The basic differential equation for the rocket motion may be written as \( m \, dv = -u \, dm \), or
\( dv = -\frac{u}{m} \, dm \). Integrating both sides we have
\[ V_f - V_i = -u \left( \ln m_f - \ln m_i \right) \]
where \( V_i \) and \( V_f \) are the initial and final velocities, and \( m_i \) and \( m_f \) the initial and final masses. Thus
\[ \Delta V = V_f - V_i = u \ln \left( \frac{m_i}{m_f} \right) \]
In the case at hand we suppose \( \Delta V = 3u \), and hence
\[ \ln \left( \frac{m_i}{m_f} \right) = 3 \]
\[ \frac{m_i}{m_f} = e^3 \] and
\[ m_0 = m_s + m_r + m_f \]
and
\[ m_f = m_s + m_r \] since
the fuel have been exhausted. Hence
\[ \frac{m_0}{m_f} = 1 + \frac{m_f - m_0}{m_s + m_r} = e^3 \]
\[ \frac{M_f}{m_0} = e^3 - 1 = 19. \]
Ukrainian technology is pretty good!
3) Brachistochrone in $1/r$ potential

a) The potential energy of a particle of mass $m$ in a gravitational potential $\Phi = -k/r$ is $m \Phi = U = -mk/r$. This plus the kinetic energy $\frac{1}{2} mv^2$ is constant, and since at $r = r_0$ the speed $v = 0$, we have $\frac{1}{2} mv^2 = mk/r = -mk/r_0$.

So $v^2 = r^2 \dot{\theta}^2 + r^2 = 2k \left( \frac{1}{\rho^2} - \frac{1}{r_0^2} \right)$. Along the curve $r = \rho(\theta)$ one has $v = [\rho'(\theta)] \dot{\theta}$, thus $\left[ \rho^2 + \rho'^2 \right] \dot{\theta}^2 = 2k \left( \frac{1}{\rho^2} - \frac{1}{r_0^2} \right) = \left( \rho^2 + \rho'^2 \right) \left( \frac{d\theta}{dt} \right)^2$

Therefore $dt/d\theta = \sqrt{\frac{\rho^2 + \rho'^2}{2k(r_0 - \rho)}} = f(\rho, \rho')$

$$T = \int_0^{\Theta} \sqrt{\frac{(\rho^2 + \rho'^2)\rho}{2k(r_0 - \rho)}} \, d\theta = \int_0^{\Theta} f(\rho, \rho') \, d\theta$$

where $\rho(\theta)$ is given and determines $\rho'(\theta)$.

b) The differential equation for the minimizing $\rho(\theta)$ is obtained from Euler's equation $\frac{\partial f}{\partial \rho} = \frac{d}{d\theta} \frac{\partial f}{\partial \rho'}$, or, since $\frac{\partial f}{\partial \theta} = 0$, one can also use $\frac{d}{d\theta} \frac{\partial f}{\partial \rho'} = \text{constant} = k$

$$\frac{\partial f}{\partial \rho'} = \rho' \sqrt{\frac{\rho_0 \rho}{2k(r_0 - \rho)(\rho^2 + \rho'^2)}}$$

So the differential equation, or its first integral, can be written as

$$\rho'^2 \sqrt{\frac{\rho_0 \rho}{2k(r_0 - \rho)(\rho^2 + \rho'^2)}} = \sqrt{\frac{(\rho^2 + \rho'^2)\rho_0}{2k(r_0 - \rho)}} - k$$
(1) One would expect to be able to use the solution of the brachistochrone problem for uniform gravity in cases in which any plausible \( r = \rho(\theta) \) remains in a domain where the gravitational potential is approximately of the form

\[
\overline{\psi}(\vec{r}) = \text{const} + g z + b \left[ \frac{x^2 + y^2}{2} - z^2 \right] + \ldots
\]

where \( x, y, z \) are measured from the point

\( \vec{r} = (r_0, \theta = 0) \) in which \( z = r - r_0, \ x = r_0 \theta \) and we set \( y = 0 \) to lowest order in \( r \) and \( \theta \). Here \( b \) will be of order \( g/r_0 \), so if \( |z| < r_0 \), say \( z = \epsilon r_0, |\epsilon| \ll 1 \),

and likewise \( x \) is of the same order — which means \( |\theta| \ll 1 \), the \( 'b' \) terms will be small compared with \( g z \).

Consequently, we expect that when

\( r_0 - r_1 < r_0, \ \ \ \theta_1 < 1 \)

the solution to the uniform-field brachistochrone problem should be adequate for the case under consideration.
Spinning disk on flat table

a) Let \( \bar{R} \) be the center of mass = center of disk, \( \bar{F}_c \) be the point of contact, \( M \bar{g} \) force of gravity, \( \bar{f}_n \) and \( \bar{f}_t \) as normal and tangential forces of table on disk. For clarity, \( M \bar{g} \) and \( \bar{f}_n \) have been slightly distorted; \( \bar{f}_n \) passes through \( \bar{R} \), \( M \bar{g} \) through \( \bar{F}_c \). The total torque on the disk will be:

\[
\bar{N} = \bar{R} \times M \bar{g} + \bar{F}_c \times \bar{f}_n + \bar{F}_c \times \bar{f}_t
\]

As the disk has zero net momentum in the upwards direction, necessarily \( M \bar{g} = -\bar{f}_n \), so

\[
\bar{R} \times M \bar{g} + \bar{F}_c \times \bar{f}_n = (\bar{R} - \bar{F}_c) \times M \bar{g} = 0 \text{ because } \bar{R} - \bar{F}_c
\]
is parallel to \( \bar{g} \). But if we compute the torques about a point on the table, \( \bar{F}_c \) will be a vector parallel to \( \bar{f}_t \), so \( \bar{F}_c \times \bar{f}_t = 0 \); about the contact point itself \( \bar{F}_c = 0 \). So \( \bar{N} = 0 \) about a point on the table, including the contact point.

The total angular momentum \( \bar{I} = \bar{R} \times M \bar{V} + \bar{I}' \), where \( \bar{I}' \) = angular momentum about center of mass, \( \bar{V} \) = velocity of center of mass. Since \( \bar{V} \) is parallel to the table, \( \bar{I}' \) is measured from any point on the table only the component perpendicular to \( \bar{V} \) enters \( \bar{R} \times \bar{V} \), i.e., \( \bar{R} \times \bar{V} = (\bar{R} - \bar{F}_c) \times \bar{V} \) is independent of which point on the table is the origin of \( \bar{R} \). Obviously \( \bar{I}' \) does not depend on the origin of coordinates, so \( \bar{I}' \) is the same about any point on the table.
6) About some point on the table top we know that \( \mathbf{L} \) must be independent of time, since \( \mathbf{\dot{L}} = \mathbf{N} = 0 \), by part (a). Initially \( L = I \omega_0 \), and after the disc has stopped slipping

\[
L = M \rho V + I \omega_1 = M \rho V + I \omega_1
\]

\[
= M \rho^2 \omega_1 + c M \rho^2 \omega_1 = (1+c) M \rho^2 \omega_1
\]

Since \( I \omega_0 = c M \rho^2 \omega_0 \), we conclude that

\[
c M \rho^2 \omega_0 = (1+c) M \rho^2 \omega_1 \implies \frac{\omega_1}{\omega_0} = \frac{c}{1+c}
\]

7) The initial kinetic energy of the cylinder is

\[
T_0 = \frac{1}{2} I \omega_0^2 = \frac{1}{2} c M \rho^2 \omega_0^2
\]

Its final kinetic energy has two terms: rotation about the center of mass plus motion of the center of mass:

\[
T_1 = \frac{1}{2} I \omega_1^2 + \frac{1}{2} MV^2 = \frac{1}{2} M \rho^2 (1+c) \omega_1^2
\]

\[
= \frac{1}{2} M \rho^2 \omega_0^2 \left[ \frac{c^2}{1+c} \right]
\]

So

\[
T_0 - T_1 = \frac{1}{2} M \rho^2 \omega_0^2 \left[ c - \frac{c^2}{1+c} \right] = \frac{1}{2} \frac{(c)}{1+c} M \rho^2 \omega_0^2
\]

is the change in kinetic energy, which decreases unless \( c = 0 \). So energy is not conserved, and this is not surprising, because there is frictional sliding going on for \( t < T \), and kinetic energy is transformed to heat.
Alpha particles on carbon target

a) Derive $T_1' / T_2' = m_2 / m_1$ for center of mass. In the center of mass system the center of mass is (by definition) not moving, so the total momentum is

$$m_1 \vec{v}_1 + m_2 \vec{v}_2 = 0$$

where $\vec{v}_1$ and $\vec{v}_2$ are the velocities of particles 1 and 2. This means their magnitudes $|\vec{v}_1| = |\vec{v}_2|$ are related by

$$v_1 = \left( \frac{m_2}{m_1} \right) v_2$$

The kinetic energies are thus related by

$$\frac{T_1'}{T_2'} = \frac{\frac{1}{2} m_1 v_1^2}{\frac{1}{2} m_2 v_2^2} = \frac{m_1}{m_2} \left( \frac{m_2}{m_1} \right)^2 \frac{v_2^2}{v_2^2} = m_2 / m_1$$

b) Kinetic energy of alpha before collision $= T_0 = \sum$ of kinetic energies after collision (collision elastic $\Rightarrow Q = 0$) $= 32 + 12 = 44$ MeV, by conservation of energy. But then

$$T_0 = \frac{1}{2} M V^2 + T_1' + T_2' = \text{center of mass energy + energy about the center of mass}$$

Now $V = \frac{m_1 u_1}{m_1 + m_2} = \text{velocity of center of mass, } u_i = \text{velocity dimensionless}$.

Thus $V = \frac{1}{1 + \frac{3}{2}} u_i = \frac{1}{4} u_i$, $M V^2 = (m_1 + m_2) V^2 = \frac{4}{3} \frac{m_1 u_1^2}{m_1 + m_2}$

so $\frac{1}{2} M V^2 = \frac{1}{4} \left( \frac{1}{2} m_1 u_1^2 \right) = 11$ MeV, and

$$T_1' + T_2' = 44 - 11 = 33 \text{ MeV}, \text{ while } T_1' / T_2' = 3, \text{ so }$$

$T_1' = 24 \frac{3}{4} \text{ MeV}, \ T_2' = 8 \frac{1}{4} \text{ MeV}$
c) Given a $Q = -8 \text{ MeV}$, with final energies 32 and 12 MeV for alpha and carbon, the initial energy of the alpha must have been

$$ T_0 = 32 + 12 + 8 = 52 \text{ MeV} $$

The center of mass term $\frac{1}{2} MV^2$ is $\frac{1}{4}$ of this or 13 MeV, using same argument from (b), so in the center of mass before the collision, we have

$$ 52 - 13 = 39 \text{ MeV} = T_1' + T_2' $$

The ratio $T_1' / T_2' = 3$ holds before or after the collision, so now

$$ T_1' = 29 \frac{1}{4} \text{ MeV}, \quad T_2' = 9 \frac{3}{4} \text{ MeV} $$

before collision.

But after the collision takes place we have

$$ T_1'' + T_2'' = 39 - 8 = 31 \text{ MeV}, \text{ so splitting this up using } T_1'' / T_2'' = 3 we get

$$ T_1'' = 23 \frac{1}{4} \text{ MeV}, \quad T_2'' = 7 \frac{3}{4} \text{ MeV} \text{ after collision.} $$