a) The Hamiltonian is the sum of the kinetic plus potential energy, thus
\[ H = \frac{p^2}{2m} + mgy. \]
Hamilton’s equations are
\[ \dot{y} = \frac{\partial H}{\partial p} = \frac{p}{m}; \quad \dot{p} = -\frac{\partial H}{\partial y} = -mg. \]
Differentiating the first and using the second gives us the expected
\[ \ddot{y} = \frac{\dot{p}}{m} = -g. \]

b) Alternative solutions to this part:
(i) \( H \) is constant along such a trajectory, and solving \( H = \frac{p^2}{2m} + mgy \) for \( y \),
\[ y = \frac{H}{mg} - \frac{p^2}{2gm^2} = \text{constant} - \frac{p^2}{2gm^2}, \]
with different \( H \) values or different constants defining different trajectories.
(ii) Integrating \( dy/dp = \dot{y}/\dot{p} = -p/gm^2 \) gives \( y = \text{constant} - \frac{p^2}{2gm^2} \).
(iii) Along a trajectory \( p = p_0 - mgt \), so \( t = (p_0 - p)/mg \). Inserting this in \( y = y_0 + (p_0/m)t - \frac{1}{2}gt^2 \) (note: \( \dot{y} = p/m = p_0/m \) at \( t = 0 \)) yields, after a little algebra, \( y = \text{constant} - \frac{p^2}{2gm^2} \).

c) The fact that \( \dot{p} = -mg \) is independent of \( y \) means that the horizontal line \( p = p_1 \) at \( t = 0 \) will at time \( t = \tau > 0 \) be shifted to a horizontal line \( p = p'_1 = p_1 - mg\tau \); similarly the \( p_2 \) line shifts to \( p'_2 = p_2 - mg\tau \), so \( \Delta p = p_2 - p_1 = p'_2 - p'_1 \) remains the same. Thus the region \( R \) gets mapped to a region \( R' \) of the same vertical height \( \Delta p \). The difference \( \Delta y \) in \( y \) values between the two trajectories at a fixed \( p \) is the difference of two constants, see (b). Therefore the area of \( R \), \( \Delta y \cdot \Delta p \), is the same as the area of \( R' \). This is what one would expect from Liouville’s theorem, which states that phase-space “volume” (in this case the “volume” is the area) of some region does not change with time as the points in the region evolve to a later time.
a) Draw a line at constant $E$, total energy, on the $V(r)$ diagram, see sketch. This intersects the $V(r)$ curve at the minimum and maximum values $r_1$ and $r_2$ of $r$, so if $r_2$ is given, this construction determines $r_1$. The reason it works is that when $r = r_1$ or $r_2$, the radial part of the kinetic energy, $\frac{1}{2}\mu \dot{r}^2$, is 0, because $\dot{r} = 0$: the orbit is at an extreme value of $r$. Thus $E = U + T = U + \frac{l^2}{2\mu r^2} = V(r)$ for these values of $r$. This construction clearly uses conservation of energy $E$, which must be the same at both $r_1$ and $r_2$, and conservation of angular momentum $l$, as this value of $l$ determines the $V(r)$ curve.

b) Since $l = \mu r^2 \dot{\theta}$ is constant (conservation of angular momentum), we know that $\dot{\theta} = l/\mu r^2$, whence it follows that

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{r_2^2}{r_1^2} = \frac{1}{\lambda^2}.$$ 

To find $T_1/T_2$, use the fact that at both $r_1$ and $r_2$ the $\dot{r}$ contribution to the kinetic energy vanishes, so $T = 0 + \frac{1}{2}\mu r^2 \dot{\theta}^2$, and

$$\frac{T_1}{T_2} = \frac{r_1^2 \dot{\theta}_1^2}{r_2^2 \dot{\theta}_2^2} = \frac{\lambda^2}{\lambda^4} = \frac{1}{\lambda^2}.$$