33-331 Physical Mechanics I. Fall Semester, 2008
Solutions to Third Hour Exam

a) The Hamiltonian is the sum of the kinetic plus potential energy, thus

$$H = p^2/2m + mgy.$$

Hamilton's equations are

$$\dot{y} = \partial H/\partial p = p/m; \quad \dot{p} = -\partial H/\partial y = -mg.$$

Differentiating the first and using the second gives us the expected

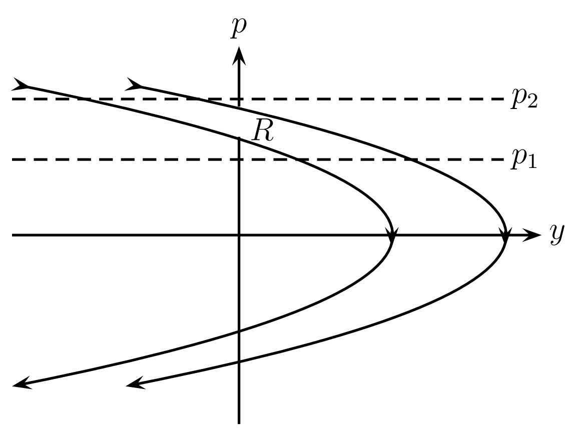
$$\ddot{y} = \dot{p}/m = -g.$$

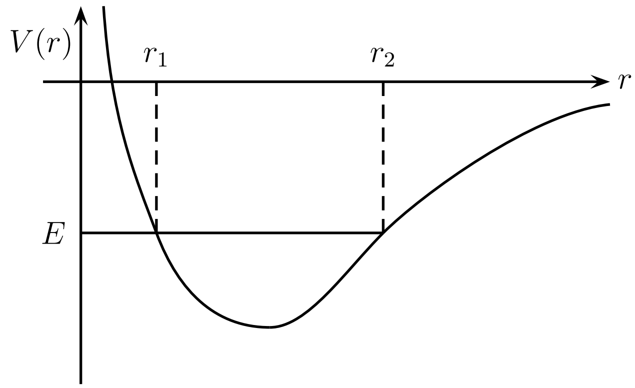
b) Alternative solutions to this part:
(i) H is constant along such a trajectory, and solving $H = p^2/2m + mgy$ for y,

$$y = H/mg - p^2/2gm^2 = \text{ constant } - p^2/2gm^2,$$

with different H values or different constants defining different trajectories.
(ii) Integrating $dy/dp = \dot{y}/\dot{p} = -p/gm^2$ gives $y = \text{ constant } - p^2/2gm^2$.
(iii) Along a trajectory $p = p_0 - mgt$, so $t = (p_0 - p)/mg$. Inserting this in $y = y_0 + (p_0/m)t - \frac{1}{2}gt^2$ (note: $\dot{y} = p/m = p_0/m$ at $t = 0$) yields, after a little algebra, $y = \text{ constant } - p^2/2gm^2$.

c) The fact that $\dot{p} = -mg$ is independent of y means that the horizontal line $p = p_1$ at $t = 0$ will at time $t = \tau > 0$ be shifted to a horizontal line $p = p_1' = p_1 - mg\tau$; similarly the p_2 line shifts to $p_2' = p_2 - mg\tau$, so $\Delta p = p_2 - p_1 = p_2' - p_1'$ remains the same. Thus the region R gets mapped to a region R' of the same vertical height Δp. The difference Δy in y values between the two trajectories at a fixed p is the difference of two constants, see (b). Therefore the area of R, $\Delta y \cdot \Delta p$, is the same as the area of R'. This is what one would expect from Liouville's theorem, which states that phase-space "volume" (in this case the "volume" is the area) of some region does not change with time as the points in the region evolve to a later time.





a) Draw a line at constant E, total energy, on the $V(r)$ diagram, see sketch. This intersects the $V(r)$ curve at the minimum and maximum values r_1 and r_2 of r, so if r_2 is given, this construction determines r_1. The reason it works is that when $r = r_1$ or r_2, the radial part of the kinetic energy, $\frac{1}{2}\mu\dot{r}^2$, is 0, because $\dot{r} = 0$: the orbit is at an extreme value of r. Thus $E = U + T = U + l^2/2\mu r^2 = V(r)$ for these values of r. This construction clearly uses conservation of energy E, which must be the same at both r_1 and r_2, and conservation of angular momentum l, as this value of l determines the $V(r)$ curve.

b) Since $l = \mu r^2\dot{\theta}$ is constant (conservation of angular momentum), we know that $\dot{\theta} = l/\mu r^2$, whence it follows that

$$\frac{\dot{\theta}_1}{\dot{\theta}_2} = \frac{r_2^2}{r_1^2} = \frac{1}{\lambda^2}.$$

To find T_1/T_2, use the fact that at both r_1 and r_2 the $\dot{r}$ contribution to the kinetic energy vanishes, so $T = 0 + \frac{1}{2}\mu r^2\dot{\theta}^2$, and

$$\frac{T_1}{T_2} = \frac{r_1^2\dot{\theta}_1^2}{r_2^2\dot{\theta}_2^2} = \frac{\lambda^2}{\lambda^4} = \frac{1}{\lambda^2}.$$