X32

33-331 Physical Mechanics Fall 2008
Solutions to Second Hour Exam

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a) The gravitational potential *outside* a spherically symmetrical mass distribution is exactly the same as if all the mass were concentrated at the center of the sphere. Consequently, since we are are interested in the potential in the $x = 0$ plane, which lies outside both massive objects, we can when calculating it replace each object with a point mass M located a distance D from the origin along the x axis. The gravitational potential of a point mass M at a distance R is $-GM/R$, and in the case at hand $R^2 = D^2 + y^2 + z^2$ for a point at $(0, y, z)$. Thus

$$\Phi(x = 0, y, z) = -2GM/\sqrt{D^2 + y^2 + z^2},$$

with the factor of 2 because each object makes the same contribution, and the potentials add (principle of superposition).

b) Near the origin $x = 0 = y = z$ we must have

$$\Phi(x, y, z) = a + Ax^2 + By^2 + Cz^2,$$

and no other terms up to quadratic order for a power series expansion of the potential. The reasons are as follows. By symmetry the potential must be unchanged by any rotation about the x axis, and by reflection in the $x = 0$, i.e., in the y, z plane. Hence a term linear in x, say αx, would change sign under a reflection that leaves Φ unchanged, and must be absent from the expansion. Similarly, terms linear in y or z are absent, by rotations about the x axis or reflections in the $z = 0$ or $y = 0$ planes. Again, terms in xy or xz change sign upon reflection in the $x = 0$ plane, and yz changes sign upon a 90° rotation about the x axis, or reflection in the $y = 0$ or $z = 0$ plane.

In addition one may note (this is not part of the question) that symmetry ensures that $B = C$ and the Poisson (Laplace) equation tells us that $\nabla^2\Phi = 0$, and therefore $A = -2B$.

c) The coefficient $A = -2B$ in (b) is negative, and B positive, on physical grounds—tides rise under the moon—and also from the expression in (a), where Φ is increasing (becoming less negative) as y and z increase. As the potential energy U of a point particle of mass m is $m\Phi$, it follows that U will *decrease* moving away from the origin along the x axis, unstable equilibrium, but *increase* along the y axis, stable equilibrium. One can also reach this conclusion by noting that a particle moving away from the origin along the x axis will feel the force from the nearer object increasing more rapidly than the force from the other decreases, so it is unstable. By contrast, a particle moving off along the positive y axis sees the sum of two attractive forces directed towards the centers of the two bodies, in the $-y$ direction, and hence is pulled back towards the origin.

From the result in (a) and using

$$\left(1 + \frac{y^2 + z^2}{D^2}\right)^{-1/2} \approx 1 - \frac{1}{2}\left(\frac{y^2 + z^2}{D^2}\right)$$

one finds that $B = GM/D^3$. The potential energy U of a particle of mass m confined to the y axis is mBy^2, so $k = 2mB$ in the oscillator formula, and the oscillation frequency will be $\omega = \sqrt{2mB/m} = \sqrt{2GM}/D^{3/2}$.

(cont. on other side)

② Two dimensional motion, $U = Ax^2 + By^2$

a) $L = T - U$; T = kinetic energy $= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$

Hence $L(x, y, \dot{x}, \dot{y}) = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - Ax^2 - By^2$

The two Euler equations are:

$$\frac{\partial L}{\partial x} = \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}, \text{ or } -2Ax = \frac{d}{dt}(m\dot{x}) = \boxed{m\ddot{x} = -2Ax}$$

$$\frac{\partial L}{\partial y} = \frac{d}{dt}\frac{\partial L}{\partial \dot{y}}, \text{ or } -2By = \frac{d}{dt}(m\dot{y}) = \boxed{m\ddot{y} = -2By}$$

b) ~~If we~~ The constraint $x^2 + y^2 = R^2$ means the particle moves on a circle of radius R centered at the origin, so we use polar coordinates and write

$x = R\cos\theta$, $\dot{x} = -R(\sin\theta)\dot{\theta}$ $y = R\sin\theta$, $\dot{y} = +R(\cos\theta)\dot{\theta}$

Inserting these into the Lagrangian of part (a) gives us

$$L = T - U = \frac{1}{2} m R^2 \dot{\theta}^2 - R^2[A\cos^2\theta + B\sin^2\theta]$$

We have one generalized coordinate θ, so one Euler equation

$$\frac{\partial L}{\partial \theta} = -2R^2[-A\cos\theta\sin\theta + B\sin\theta\cos\theta]$$

$$= 2R^2(A - B)\cos\theta\sin\theta = R^2(A - B)\sin 2\theta$$

$\partial L/\partial\dot{\theta} = mR^2\dot{\theta}$. So $\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$ gives us

$$\boxed{m\ddot{\theta} = 2(A - B)\cos\theta\sin\theta = (A - B)\sin 2\theta}$$