13 Mechanical Waves Fall 2003

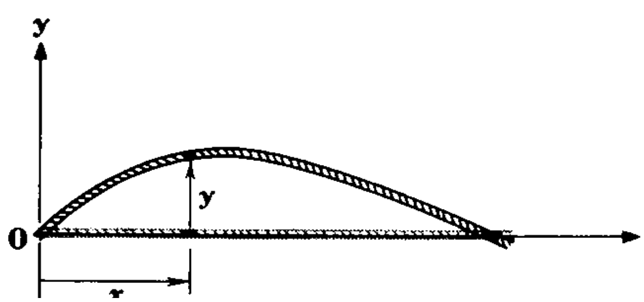
A *wave* is a disturbance from an equilibrium state that moves or *propagates* from one region of space to another. Wave phenomena are found in all areas of physics. The wave concept plays a central and overwhelmingly important role in all of physical theory and is a key unifying element in the most diverse branches of physics.

Familiar examples include waves on the surface of a liquid, sound waves (a periodic disturbance from a state of uniform pressure), and electromagnetic waves (the passage of time-varying electromagnetic field patterns through otherwise empty space). Mechanical waves always are associated with a *wave medium*, such as air, a solid material, or a liquid surface. In a *longitudinal* wave, the molecules of the medium move back and forth *parallel* to the direction of the travel of the wave during wave propagation. In a *transverse* wave, they are displaced along a line *perpendicular* to the wave's direction of travel.

In air, sound waves are longitudinal; in solids and liquids they can be either longitudinal or transverse. Electromagnetic waves have no mechanical medium, but in regions far from the source the electric and magnetic fields are perpendicular to the direction of propagation. Thus electromagnetic waves are classified as transverse.

Waves on a Stretched Rope

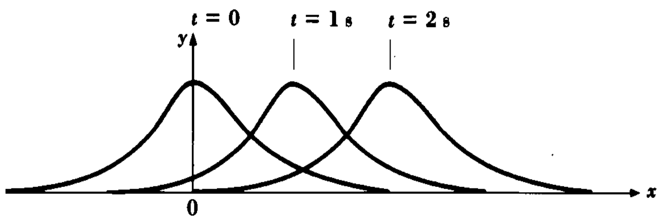
One of the simplest kinds of mechanical wave to visualize and analyze is wave motion on a stretched rope or string. We'll use this system to illustrate several of the most important features of wave propagation. Suppose we tie one end of a long rope to a stationary point, stretch the rope out horizontally (neglecting any sag due to gravity), and then give the end we are holding a back-and-forth transverse motion. The result is a *wave pulse* that travels along the length of the rope. Observation shows that the pulse travels with a definite speed, maintaining its shape as it travels, and that the individual particles making up the rope move back and forth in a direction *perpendicular* to the rope's equilibrium position, not parallel to it. Thus the wave is *transverse*.



To analyze waves on a stretched rope in detail, we'll use the coordinate system shown. The equilibrium position is along the x-axis, and the transverse displacement of any point away from this position is y. Thus y is a function of both x (the undisplaced position of the point) and time t; $y = f(x, t)$. This is called the *wave function*; if we know the wave function for a particular wave motion, we know everything there is to know about the motion of the system. We'll explore this remark in detail later.

Example: Suppose a transverse wave pulse on a rope is described at time $t = 0$ by the equation

$$y = \frac{1}{1+x^2}, \qquad \text{where } x \text{ and } y \text{ are both measured in meters.}$$



The crest of the pulse (the maximum value of y) is at $x = 0$. Suppose the pulse moves in the $+x$ direction (i.e., the direction of increasing x) with a constant speed of 2 m/s. Then at the later time $t = 1$ s, the crest of the pulse has moved to $x = 2$ m, and the corresponding function is

$$y = \frac{1}{1+(x-2\text{ m})^2}. \qquad \text{At time } t = 2\text{ s}, \quad y = \frac{1}{1+(x-4\text{ m})^2}, \quad \text{and so on.}$$

Generalizing, we see that if the speed of propagation of the wave is denoted by c, then at any time t the shape of the wave pulse is given by

$$y = \frac{1}{1+(x-ct)^2}.$$ That is, in time t the pulse has traveled a distance ct. If you run alongside the rope with speed c, the quantity $(x - ct)$ at your moving location is constant, and your speed is the same as that of the wave pulse

More generally, any function of x and t that contains the variables x and t only in the combination $(x - ct)$ represents a wave traveling in the direction of increasing x with wave speed c. Two simple examples (the first a pulse, the second a sinusoidal wave) are

$$e^{-k(x-ct)^2} \qquad \text{and} \qquad \cos k(x-ct). \tag{1}$$

A pulse may also originate at some point to the right of the origin and travel in the direction of *decreasing* x. In that case we replace the quantity $(x - ct)$ in all the above expressions by $(x + ct)$. Any function containing x and t only in this specific combination represents a wave traveling in the $-x$ direction with speed c.

We can show that *any* function $y = f(x, t)$ that contains x and t only in the combination $(x - ct)$ or $(x + ct)$, or any linear combination of such functions, must satisfy the partial differential equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}. \tag{2}$$

Thus any wave function for a wave traveling in *either* the $+x$ or $-x$ direction, or any linear combination of such functions, must satisfy this equation, one of several forms of the *wave equation*. Proof of this statement is left as a problem.

Periodic Waves

A particularly important class of waves is one in which each particle of the medium undergoes a *periodic* motion with period T (the time for one cycle) and frequency $f = 1/T$ (the number of cycles per unit time). In this case the pattern of the wave motion on the rope is also *spatially* periodic. During one period, the wave travels a distance cT, so the displacement of the rope consists of a series of identical patterns, each with length cT. This length is called the *wavelength* of the wave, denoted by λ. During a time equal to T, the wave travels a distance λ. Thus we have the relations

$$\lambda = cT = \frac{c}{f} \qquad \text{or} \qquad c = \lambda f. \tag{3}$$

Of particular interest are *sinusoidal* waves, in which the position of each particle varies sinusoidally with time, i.e., with simple harmonic motion. Then at each value of x, the displacement y of a point on the string is a sinusoidal function of time. And at any time t, if we take a picture of the instantaneous shape of the string, we find that y varies sinusoidally with x.

Here's a way to devise a wave function for a sinusoidal wave. We give the end of the rope (at $x = 0$) a sinusoidal motion $y = A \cos \omega t$, where as usual $\omega = 2\pi f$ is the angular frequency of the motion and A is its amplitude. Then every other point on the rope also moves with sinusoidal motion, with the same frequency and period but with a phase lag that is proportional to the distance x from the end. That is, at point x,

$$y = A\cos(\omega t - \varphi). \tag{4}$$

If x is exactly one wavelength $(x = \lambda)$, the phase lag φ is exactly $\varphi = 2\pi$ (i.e., one cycle). If $x = \lambda/2$, $\varphi = \pi$, and so on. In general, at *any* point x the phase lag is $\varphi = 2\pi x/\lambda$, and the general wave function is

$$y(x,t) = A\cos\left(\omega t - 2\pi \frac{x}{\lambda}\right). \tag{5}$$

Using the identity $\cos(\alpha) = \cos(-\alpha)$ and the relations $\omega = 2\pi f = 2\pi/T$, we can reverse the order of terms and write this in the more customary forms

$$y(x,t) = A\cos\left(2\pi\frac{x}{\lambda} - \omega t\right) = A\cos\left[2\pi\left(\frac{x}{\lambda} - \frac{t}{T}\right)\right]. \tag{6}$$

The second form shows explicitly that when x increases by one wavelength (λ) the cosine function goes through one period (2π), and that when t increases by one period (T) the cosine function again goes through one period.

It is often convenient to express some of the above relations in terms of a quantity k called the *wave number* or the *propagation constant*, defined as

$$k = \frac{2\pi}{\lambda}. \tag{7}$$

Using this notation, we can rewrite Eq. (3) as

$$\frac{2\pi}{k} = \frac{c}{\omega/2\pi}, \qquad \text{or} \qquad \omega = ck. \tag{8}$$

We can also re-express Eq. (6) in any of the following forms:

$$\cos\left[2\pi\left(\frac{x}{\lambda} - f\,t\right)\right], \quad \cos\left[2\pi\,f\left(\frac{x}{c} - t\right)\right], \quad \cos(k\,x - \omega t), \quad \cos\left[\omega\left(\frac{x}{c} - t\right)\right]. \qquad (9)$$

Each of these forms can be expressed as a function of the quantity $u = x - ct$. Proof of these statements is left as a problem. Similar expressions can be written for sinusoidal waves traveling in the $-x$ direction; all the $-$'s in Eqs. (6) and (9) become $+$'s.

The wave function $y(x, t)$ gives the transverse displacement (y) at any time t of a point x. The transverse *velocity* v_y of the point is the time rate of change of y with respect to t. Because y is also a function of x, we write this relationship using a *partial* derivative:

$$v_y = \frac{\partial y}{\partial t}. \qquad (10)$$

Also, the *slope* M of the rope at any point is the partial derivative of y with respect to x:

$$M = \frac{\partial y}{\partial x}. \qquad (11)$$

Example: Suppose the wave function is $y(x, t) = A\cos(kx - \omega t)$. The transverse speed v_y of a point on the rope is given by

$$v_y = \frac{\partial y}{\partial t} = A\omega\sin(kx - \omega t).$$

For example, at the point $x = 0$,

$$y = A\cos(-\omega t) = A\cos(\omega t) \qquad \text{and} \qquad v_y = A\omega\sin(-\omega t) = -A\omega\sin(\omega t)$$

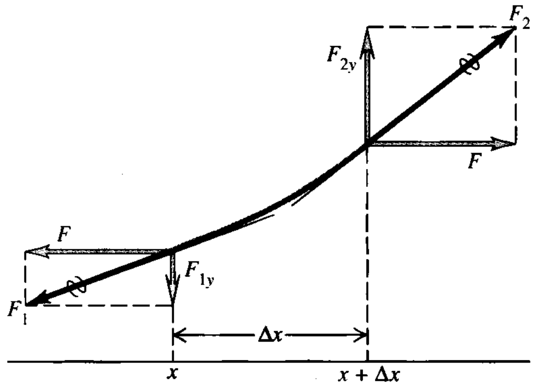
At time $t = 0$, the point has displacement A and is instantaneously at rest. At a slightly later time, y is a little less than A, and the velocity has become slightly negative (as the point moves downward toward its equilibrium position).

Speed of Waves on a Rope

The speed of wave propagation on a stretched rope is determined by its mechanical properties; these are the tension F and the mass per unit length μ (also called the *linear mass densit*y). It turns out that the wave speed c is given by

$$c = \sqrt{\frac{F}{\mu}}. \qquad (12)$$

We'll derive this relationship, but first we note that it is intuitively reasonable. Intuition suggests that waves should travel more slowly on a more massive rope than on a lighter one, and that greater tension should lead to a greater wave speed.



To derive Eq. (12), we apply Newton's second law to a short segment of string that in the equilibrium position would have length Δx. (The length in any displaced position is somewhat greater, as shown in the diagram, where vertical displacements are greatly exaggerated.) The mass of this segment is $\mu \Delta x$.

The forces acting at the two ends are shown. Because the motion is assumed to be transverse to the direction of propagation, the segment has no component of acceleration in the x direction, so the total horizontal force acting on it must be zero. The magnitude of the x component of force on each direction is just equal to the tension F.

To find the y components of force at the two ends, we note that at each end the ratio of vertical to horizontal components is equal (apart from sign) to the slope of the rope at each point. Taking signs into account, we have

$$\frac{F_{1y}}{F} = -\left(\frac{\partial y}{\partial x}\right)_x, \qquad \frac{F_{2y}}{F} = \left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} \tag{13}$$

Now we apply Newton's second law; equating the *net* y component of force to the mass $\mu \Delta x$ of the segment times its transverse acceleration:

$$F_{2y} + F_{1y} = F\left[\left(\frac{\partial y}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right] = (\mu \Delta x)\frac{\partial^2 y}{\partial t^2}. \tag{14}$$

Finally, we divide both sides by Δx and take the limit as $\Delta x \to 0$. In this limit the left side becomes the *second* derivative of y with respect to x, and the final result is

$$F\frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2}, \qquad \text{or} \qquad \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{F}\frac{\partial^2 y}{\partial t^2} \tag{15}$$

We conclude that the wave function $y(x, t)$ for any wave motion that is consistent with Newton's second law must satisfy Eq. (15). But we have also seen previously that the wave function must satisfy Eq. (2). Comparing Eqs. (2) and (15), we see that both equations can be satisfied at once only if

$$\frac{1}{c^2} = \frac{\mu}{F} \qquad \text{or} \qquad c = \sqrt{\frac{F}{\mu}}, \tag{16}$$

as we asserted with Eq. (12). Thus Eq.(16) shows how the speed c of the wave is determined by the mechanical properties μ and F of the rope.

Here's a useful by-product of this derivation At any point x on the rope, the portion to the *right* of the point exerts, on the portion to the *left*, a longitudinal component of force with magnitude F (the tension) and a transverse (y component) of force F_y given by

$$F_y = F\frac{\partial y}{\partial x}. \tag{17}$$

Simultaneously, the portion on the left exerts a transverse force on the portion to the right, given by the negative of this expression (according to Newton's third law).

Reflection, Superposition, and Standing Waves

Suppose a wave pulse is initiated at the positive-x end of a stretched rope and travels in the $-x$ direction toward $x = 0$. We'll call this the *incident* pulse. Now suppose the point $x = 0$ is held stationary by a clamp, so that for any time t, $y(0, t) = 0$. What happens?

Observation shows that a second wave pulse, inverted compared to the incident pulse, originates at $x = 0$ and travels in the $+x$ direction. We'll call this the *reflected* pulse. As the incident pulse arrives at $x = 0$, it exerts a varying force on the clamp at the stationary point. By Newton's third law, that point exerts at each instant an equal and opposite force on the rope. This reaction force generates the reflected pulse.

If y_1 is the wave function for the incident pulse and y_2 is the wave function for the reflected pulse, then each of these functions separately satisfies the wave equation, Eq. (15). Because that equation is a *linear* equation, any linear combination of solutions is also a solution. This is the *principle of linear superposition*. The individual wave functions, y_1 and y_2, aren't necessarily zero at all times at the stationary point $x = 0$, but their *sum* must be zero at $x = 0$ at all times. This kind of condition is called a *boundary condition*.

Now suppose the incoming wave is a *sinusoidal* wave rather than a wave pulse. Specifically, let's assume that the incident sinusoidal wave has the wave function

$$y_1 = A\cos(kx + \omega t). \tag{18}$$

We assert that in order to satisfy the boundary condition at $x = 0$ at all times, the wave function y_2 for the reflected wave must be

$$y_2 = -A\cos(kx - \omega t) \tag{19}$$

The first $(-)$ shows that the reflected wave is *inverted* with respect to the incident wave. The *total* wave function for the system, the sum of incident and reflected waves, is

$$y = y_1 + y_2 = A\cos(kx + \omega t) - A\cos(kx - \omega t). \tag{20}$$

At the stationary point $x = 0$, this becomes

$$y(0, t) = A\cos(\omega t) - A\cos(-\omega t) = 0\,.$$

This is zero because for any α, $\cos\alpha = \cos(-\alpha)$. Thus we confirm that the total wave function, Eq. (20), *does* satisfy the boundary condition that $y(0, t) = 0$ for all t.

We can gain further insight by expanding the cosine functions in Eq. (20) using the identities

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta. \tag{21}$$

We leave the details of this calculation as an exercise; the result is

$$y = y_1 + y_2 = -2A\sin kx\,\sin\omega t = (-2A\sin kx)\sin\omega t. \tag{22}$$

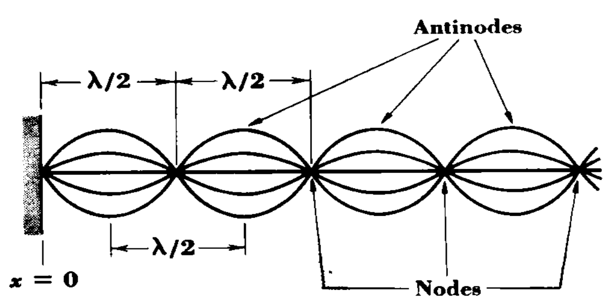
This expression *does not* contain the variables x and t in the combination $x - ct$ or $x + ct$, and it doesn't represent a wave traveling in one direction or the other. Instead, the displacement of every particle is proportional to $\sin\omega t$, so the particles all move *in phase* (or 1/2 cycle out of phase) with angular frequency ω. The amplitude of motion of each particle is (apart from sign) $2A\sin kx$. Thus the appearance is that of a sinusoidal shape that doesn't move along the length of the string but grows larger and smaller with time. Such a wave is called *a standing wave.*

Some particles never move at all. At points where kx is an integer multiple of π, that is

$$kx = n\pi \qquad (n = 1,\ 2,\ 3,\ \cdots\ ),$$

$\sin kx = 0$. Such points are called *node* points or *nodes*. They are located at values of x such that

$$x = n\frac{\pi}{k} = n\frac{\lambda}{2}. \tag{23}$$



The node points are equally spaced, a half-wavelength apart, along the length of the rope. The point $x = 0$ is of course a node point.

Midway between each two adjacent nodes are points of maximum displacement, with amplitude $2A$. These points, called *antinodes*, are located at values of x given by

$$kx = (n + \tfrac{1}{2})\pi \qquad \text{or} \qquad x = (n + \tfrac{1}{2})\frac{\pi}{k} = (n + \tfrac{1}{2})\frac{\lambda}{2} \tag{24}$$

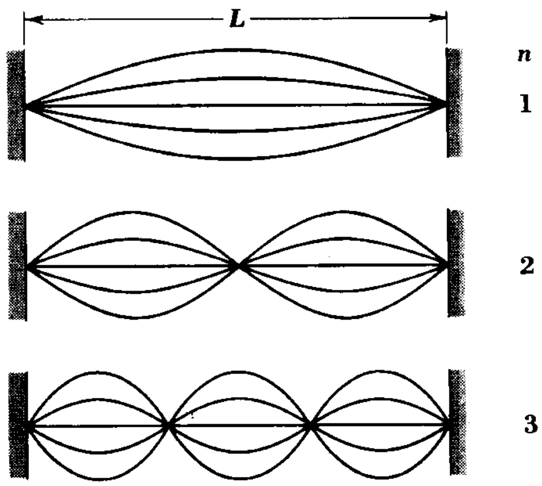
Like the nodes, the antinodes are spaced one-half wavelength apart.

Now suppose the rope has finite length L and that *both* ends are held stationary. Then the points $x = 0$ and $x = L$ must both be nodes. Because the nodes must be spaced a half-wavelength apart, this condition can be satisfied only if L is an integer number of half-wavelengths. That is, a standing wave on a rope of length L, with both ends held, is possible only for certain wavelengths and therefore only for certain frequencies. The possible wavelengths, which we denote by λ_n, are given by

$$L = n\frac{\lambda_n}{2} = n\frac{\pi}{k} \qquad \text{or} \qquad \lambda_n = \frac{2L}{n} \qquad (n = 1,\ 2,\ 3,\ \cdots). \tag{25}$$

The corresponding permitted frequencies and angular frequencies, from $c = \lambda f$, are

$$f_n = n\frac{c}{2L}, \qquad \omega_n = 2\pi f_n = n\frac{\pi c}{L} \qquad (n = 1,\ 2,\ 3,\ \cdots\ ), \tag{26}$$



This result shows that the lowest-frequency standing wave has frequency $c/2L$ and that all the others are integer multiples of this value. The possible frequencies are said to form a *harmonic series*,

$$f_1 = \frac{c}{2L}, \quad f_2 = \frac{2c}{2L}, \quad f_3 = \frac{3c}{2L}, \quad \cdots . \tag{27}$$

The smallest or *fundamental* frequency is f_1; all the others are *overtones* or *harmonics*. The fundamental frequency can also be expressed in terms of the mechanical properties of the rope, using Eq. (12). We invite you to show that

$$f_1 = \frac{1}{2L}\sqrt{\frac{F}{\mu}}. \tag{28}$$

This relation is the main determinant of the *pitch* of stringed musical instruments.

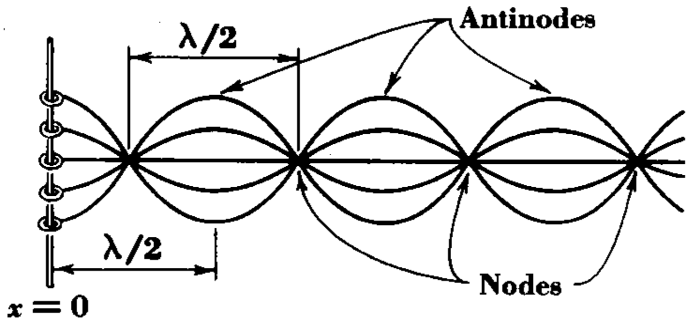
Recalling the definition of a *normal mode* of a vibrating system, we see that each value of f and the corresponding vibration pattern constitute a normal mode for this system. All particles vibrate sinusoidally with the same frequency, and there is a definite vibration pattern relating the motions of the various points. But, unlike systems containing a few springs and masses (and only a few normal modes), this system has an *infinite* number of normal modes, one for each value of the integer index n. For each mode, n is the number of antinodes, and $(n + 1)$ is the number of nodes (including the end points).

From Eq. (24), the wave number k_n for mode n is

$$k_n = \frac{n\pi}{L} \tag{29}$$

Finally, we can construct a wave function y_n for normal mode n. We begin with Eq. (22), with one change. In Eq. (22), A was the amplitude of each individual traveling wave. It is usually more convenient to make A in Eq. (22) the amplitude of the standing wave. We simply replace $(-2A)$ by A and make the appropriate substitutions for k and ω, from Eqs. (26) and (29). The final result is

$$y_n = A\sin\left(n\frac{\pi}{L}x\right)\sin\left(n\frac{\pi c}{L}t\right). \tag{30}$$



The conditions $y(0, t)$ and $y(L, t)$ (i.e., that the ends of the string at $x = 0$ and $x = L$ never move) are called *boundary conditions* for the system. Other, different boundary conditions are also possible. Suppose instead of tying the rope to a fixed point at $x = 0$, we tie it to a ring that slides without friction on a rod oriented perpendicular to the x-axis. In this case there is no transverse force on the end of the rope, and Eq. (17) requires that $\partial y/\partial x = 0$ at this point. This end is then an *antinode* rather than a node. An incoming wave is then reflected without inversion. If the incoming wave is sinusoidal, the total wave function is

$$y(x,t) = A\cos(kx + \omega t) + A\cos(kx - \omega t). \tag{31}$$

We can again use the cosine sum identity, Eq. (21), and replace $(2A)$ by A as before, to rewrite this as

$$y(x,t) = A\cos\omega t \cos kx = (A\cos kx)\cos\omega t\,. \tag{32}$$

If *both* ends of the rope are anchored to rings, as described above, then both ends are antinodes. We invite you to prove that in this case the possible frequencies are given by Eq. (27) but with the positions of the nodes and antinodes interchanged compared to the figure on page 13-8. Finally, what happens when one end is tied to a sliding ring and the other end is stationary? The answer to this question is left as an exercise.

Partial Reflection

Suppose two pieces of rope with different linear mass densities μ_1 and μ_2 are tied together and stretched, so the tension F is the same in both. Again let the equilibrium position be the x-axis, neglect any sag due to gravity, and place the knot (assumed massless) at the point $x = 0$. Now suppose a sinusoidal wave originates in the left $(x < 0)$ side of the rope and travels in the $+x$ direction. What happens?

Experiment shows that there is a *reflected* wave in the $x < 0$ region, and also a *transmitted* wave that goes into the $x > 0$ region. We'll denote the total wave function in the $x < 0$ region as y_-, and the wave function in the $x > 0$ region as y_+. If we are given the amplitude and angular frequency of the incoming wave, can we determine these quantities for the reflected and transmitted waves?

The first step is to identify the *boundary conditions* that must be satisfied at the knot (the point $x = 0$). First, the rope must be continuous at this point, so at $x = 0$, $y_- = y_+$:

$$y_-(0,t) = y_+(0,t). \tag{33}$$

Continuity of the rope also requires that the *angular frequency* of motion ω must be the same in the two sides; otherwise this condition couldn't be satisfied at all times.

Second, each section of rope exerts a transverse force at the knot, given by Eq. (17). The *total* transverse force exerted on the knot by both ropes must, according to Newton's second law, equal its mass times its acceleration. But we have assumed the knot is massless; therefore the total force must be zero. Since the tension is the same on both sides, the slopes at the point $x = 0$ also must be the same.

$$\left(\frac{\partial y_-}{\partial x}\right)_{x=0} = \left(\frac{\partial y_+}{\partial x}\right)_{x=0}. \tag{34}$$

With all these considerations in mind, we try a solution in the form

$$y_- = A\cos(k_1 x - \omega t) + B\cos(k_1 x + \omega t). \tag{35}$$

$$y_+ = C\cos(k_2 x - \omega t).$$

In these equations, A is the amplitude of the incident wave, B the amplitude of the reflected wave, (both in the region $x < 0$), and C the amplitude of the transmitted wave. (in the region $x > 0$). The angular frequency ω is the same in all functions, but the wave number k is different $(k_1,\ k_2)$ on the two sides because the linear mass densities $(\mu_1,\ \mu_2)$, and therefore the wave speeds $(c_1,\ c_2)$, are different in the two sections of rope.

Considering the first boundary condition, we evaluate Eqs. (35) at $x = 0$ and substitute the results into Eq. (33):

$$A\cos(-\omega t) + B\cos(\omega t) = C\cos(-\omega t). \qquad \text{Or, since} \qquad \cos(-\alpha) = \cos(\alpha),$$

$$A + B = C. \tag{36}$$

For the second condition, we take the derivatives of Eqs. (35) indicated in Eq. (34) and evaluate the results at $x = 0$:

$$-k_1 A\sin(-\omega t) - k_1 B\sin(\omega t) = -k_2 C\sin(-\omega t). \qquad \text{Or, since} \qquad \sin(-\alpha) = -\sin(\alpha),$$

$$k_1(A - B) = k_2 C. \tag{37}$$

Now, assuming the amplitude A of the incident wave is known, we can solve Eqs. (36) and (37) simultaneously for B and C. We leave the details as a problem; the results are

$$B = \frac{k_1 - k_2}{k_1 + k_2}A, \qquad C = \frac{2k_1}{k_1 + k_2}A. \tag{38}$$

We note that if $k_1 = k_2$, then $B = 0$ and $C = 1$. Then there is no reflected wave, and the transmitted wave has the same amplitude as the incident wave, both reasonable results.

We can re-write Eqs. (38) in terms of the wave speeds c_1 and c_2 in the two sections of rope, using the relations $\omega = c_1 k_1 = c_2 k_2$. Again we leave the details as an exercise; the results are

$$B = \frac{c_2 - c_1}{c_2 + c_1}A, \qquad C = \frac{2c_2}{c_1 + c_2}A. \tag{39}$$

Energy in Wave Motion

Every wave motion has energy associated with it, and waves can convey energy from one region of space to another. We'll explore these concepts in the context of waves on a stretched rope or string.

Considering a small segment of rope with length (in its equilibrium position) Δx, we see that the kinetic energy of the segment is

$$K = \frac{1}{2} m v^2 = \frac{1}{2} (\mu \, \Delta x) \left(\frac{\partial y}{\partial t} \right)^2 = \frac{1}{2} \mu \left(\frac{\partial y}{\partial t} \right)^2 \Delta x . \tag{40}$$

The kinetic energy *per unit length* Δx is

$$\frac{1}{2} \mu \left(\frac{\partial y}{\partial t} \right)^2 \tag{41}$$

The segment also has *potential* energy because work is required to displace and deform it from its equilibrium state. Suppose the segment is initially horizontal, at $y = y_1$, and then the right end is displaced a distance $\Delta y = \left(\frac{\partial y}{\partial x} \right) \Delta x$. After this displacement, the force acting at the right end has a transverse component F_y with magnitude $F \frac{\partial y}{\partial x}$.

The *average* transverse component of force during the displacement is half of this:

$\left(F_y \right)_{\text{ave}} = \frac{1}{2} F \frac{\partial y}{\partial x}$. The *work* W done by F_y during the displacement is

$$W = \left(F_y \right)_{\text{ave}} \Delta y = \left[\frac{1}{2} F \frac{\partial y}{\partial x} \right] \left[\frac{\partial y}{\partial x} \Delta x \right] = \frac{1}{2} F \left(\frac{\partial y}{\partial x} \right)^2 \Delta x \tag{42}$$

This is equal to the potential energy V of the segment Δx. The potential energy *per unit length* is

$$\frac{1}{2} F \left(\frac{\partial y}{\partial x} \right)^2 . \tag{43}$$

Finally, the *total* energy (kinetic plus potential) of the segment Δx is

$$E = \left[\frac{1}{2} \mu \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} F \left(\frac{\partial y}{\partial x} \right)^2 \right] \Delta x. \tag{44}$$

To find the total energy of the entire rope, we integrate Eq. (44) on x over the length of the rope. For a rope with ends at $x = 0$ and $x = L$, the total energy is

$$E = \int_0^L \left[\frac{1}{2} \mu \left(\frac{\partial y}{\partial t} \right)^2 + \frac{1}{2} F \left(\frac{\partial y}{\partial x} \right)^2 \right] dx . \tag{45}$$

Wave motion on a rope can transfer energy from one region of the rope to another. Consider a point Q on the rope. The portion of rope to the left (i.e., smaller x) of Q exerts a transverse force F_y on the portion to the right of Q (larger x). According to Eq. (17), this force is given by

$$y = -F\frac{\partial y}{\partial x}. \qquad (46)$$

As the point moves transversely, this force does work on the portion to the right of Q. The *power* P (time rate of doing work) associated with this work is given by

$$P = F_y v_y = -F\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}. \qquad (47)$$

Thus there is a flow of energy in the $+x$ direction with corresponding power (time rate of transfer of energy) given by Eq. (47).

<u>Example:</u> Derive an expression for the rate of energy flow past a given point in a rope when the wave function is $y = A\cos(kx - \omega t)$.

The derivatives in Eq. (47) are

$$\frac{\partial y}{\partial x} = -Ak\sin(kx - \omega t), \qquad \frac{\partial y}{\partial t} = A\omega\sin(kx - \omega t).$$

We substitute these into Eq. (47) and combine factors to obtain

$$P = -F\left[-Ak\sin(kx - \omega t)\right]\left[A\omega\sin(kx - \omega t)\right] \quad \text{and}$$

$$P = Fk\omega A^2\sin^2(kx - \omega t). \qquad (48)$$

Several aspects of Eq. (48) are noteworthy. First, the expression is never negative; the flow of energy is always in the $+x$ direction. Second, the energy flow rate is proportional to the *square* of the amplitude A. Finally, because $k = \omega/c$, it is proportional also to the *square* of the angular frequency ω.

The factor Fkw in Eq. (48) can be transformed into a more generally useful form by use of the relations $\omega = ck$ and $c^2 = F/\mu$, which are Eqs. (8) and (12), respectively. We get

$$Fk\omega = (\mu c^2)\left(\frac{\omega}{c}\right)\omega = \mu c\omega^2 = \mu\sqrt{\frac{F}{\mu}}\omega^2, \qquad \text{and finally}$$

$$P = \sqrt{\mu F}\,\omega^2 A^2\sin^2(kx - \omega t) \qquad (49)$$

At any given point on the rope, the *average* value of $\sin^2(kx - wt)$ over one cycle (or any integer number of cycles) is 1/2. Thus the *average* rate of energy transmission is

$$P_{\text{ave}} = \frac{1}{2}\sqrt{\mu F}\,\omega^2 A^2. \qquad (50)$$

We note that P depends only on ω, A, and the mechanical properties μ and F of the rope. The quantity $\sqrt{\mu F}$ is called the *characteristic impedance* of the rope.

Complex Exponential Functions

Calculations with sinusoidal functions can often be simplified by expressing them in terms of exponential functions with imaginary or complex arguments. The relation of sinusoidal and exponential functions is explored in Section 12, Eqs. (8) through (11). We quote here some results of that discussion.

Any complex number or function z can be expressed in the form $z = x + iy$, where x and y are *real* numbers or functions and $i = \sqrt{-1}$. The exponential function e^z is given by

$$e^z = e^x\left(\cos y + i \sin y\right). \tag{51}$$

In particular, when $x = 0$, this becomes

$$e^{iy} = \cos y + i \sin y. \tag{52}$$

This relation is called *Euler's formula.*

The real part of a complex quantity z is often denoted by $\mathrm{Re}[z]$, and the imaginary part by $\mathrm{Im}[z]$. Thus Eq. (52) can be expressed as $\mathrm{Re}[e^{iy}] = \cos y$, $\mathrm{Im}[e^{iy}] = \sin y$.

Euler's formula shows that the wave function $y(x,t) = A\cos(kx - \omega t)$ for a sinusoidal wave traveling in the $+x$ direction can be expressed as the real part of the function $Ae^{i(kx-\omega t)}$. Similarly, the function $y(x,t) = A\sin(kx - \omega t)$ is the imaginary part of $Ae^{i(kx-\omega t)}$. A sinusoidal wave traveling in the $-x$ direction can be expressed as $Ae^{-i(kx+\omega t)}$. (We include the $-$ sign in the exponent so that all the exponential functions will have the same dependence on t, contained in the factor $e^{-i\omega t}$.)

Of course, the displacements of points on a rope are always *real* quantities, but we can describe them conveniently as the real parts of complex functions.

Example: Consider again the problem of reflection of a sinusoidal wave at a boundary (at the point $x = 0$) between two sections of rope with different linear mass densities, as discussed on pages 9 and 10. Let the wave functions on the two sides of the junction be

$$y_- = Ae^{i(k_1x-\omega t)} + Be^{-i(k_1x+\omega t)}, \qquad y_+ = Ce^{i(k_2x-\omega t)} \tag{53}$$

The boundary conditions at $x = 0$ are given by Eqs. (33) and (34). We note that taking the derivatives of these functions with respect to x amounts to simply multiplying each by a factor (ik) or $(-ik)$. Applying these boundary conditions, we again obtain Eqs. (36) and (37):

$$A + B = C \qquad \text{and} \qquad k_1A - k_1b = k_2C.$$

Beats, Dispersion, and Group Velocity

When two or more sinusoidal functions having different frequencies are superimposed, interesting new features appear. To introduce the topic, we consider a stretched rope with an end at the point $x = 0$. We give this point a transverse motion that is a superposition of two sinusoidal motions with equal amplitudes but slightly different frequencies ω_1 and ω_2, as described by the expression

$$y(0,t) = A\cos(\omega_1 t) + A\cos(\omega_2 t) \tag{54}$$

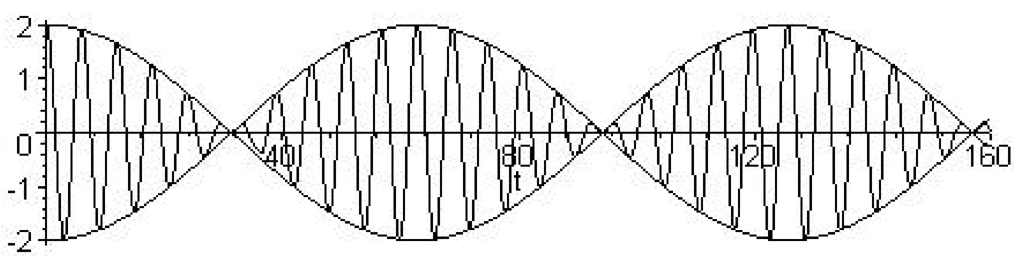
This expression is more interesting when we rewrite it in terms of the *average* ω_o of the two frequencies, and the amount $\Delta\omega$ by which each differs from the average. That is,

$$\omega_1 = \omega_o + \Delta\omega, \qquad \omega_2 = \omega_o - \Delta\omega . \tag{55}$$

(We've assumed that $\omega_1 > \omega_2$.) Now we substitute these expressions back into Eqs. (54) and expand each of the cosine functions using the cosine-sum identities. Two of the four terms subtract out, the other two add, and the final result is

$$y(0,t) = \left[2A\cos(\Delta\omega t)\right]\cos(\omega_o t) \tag{56}$$

Assuming $\Delta\omega$ is much smaller than ω_o, we can think of Eq. (56) as representing a sinusoidal motion with angular frequency ω_o and an amplitude (the quantity in square brackets) that is not constant but that varies slowly with time (with angular frequency $\Delta\omega$) between zero and $\pm 2A$. Here is a graph of Eq. (56) (displacement as a function of time) for the case $\Delta\omega = \omega_o/10$.



The figure shows that the two sinusoidal functions start out in phase at time $t = 0$, and the total amplitude is $2A$. As time goes on, one function oscillates with slightly greater frequency than the other, and the phase difference increases successively. When the phase difference reaches 1/2 cycle, there is complete cancellation. After another equal time interval, they are back in phase and the amplitude is again $2A$.

The solid curves in the figure correspond to the factor in square brackets in Eq. (56), and its negative; they constitute the *envelope* of the rapidly oscillating curve.

If the two sinusoidal functions in Eq. (54) are two sound waves, perhaps produced by two slightly out-of-tune organ pipes, the listener hears a tone with angular frequency ω_o that grows louder and softer, or *beats*, with angular frequency $2\Delta\omega = \omega_1 - \omega_2$, called the *beat frequency*. The factor of 2 results from the fact that the amplitude reaches maximum *magnitude* twice for each cycle of the function $\cos(\Delta\omega t)$; the ear hears only the *magnitude* of the amplitude variation. Thus the beat frequency is $\omega_1 - \omega_2$. Listening for beats (or their absence) is the principal means of tuning pipe organs and many other musical instruments.

Now let's consider a wave on a rope that is produced by giving the end at $x = 0$ the motion described by Eq. (54). We'll assume for now that the wave speed c is the same for all frequencies; later we'll explore what happens when the speeds of the two waves are different. The first term in Eq. (54) produces a sinusoidal wave given by

$$y_1 = A\cos(k_1 x - \omega_1 t), \qquad \text{where} \quad k_1 = \omega_1/c. \tag{57}$$

The wave function for the second term in Eq. (54) is obtained similarly, and the total wave function (from the principle of linear superposition) is

$$y = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t) \tag{58}$$

As in Eq. (55), we introduce the quantities k_o and Δk, defined by the equations

$$k_1 = k_o + \Delta k \qquad \text{and} \qquad k_2 = k_o - \Delta k. \tag{59}$$

We substitute these expressions into Eq. (58), re-group the terms, and expand the cosine functions using the cosine-sum identities:

$$\begin{aligned} y &= A\cos\big[(k_o + \Delta k)x - (\omega_o + \Delta\omega)t\big] + A\cos\big[(k_o - \Delta k)x - (\omega_o - \Delta\omega)t\big] \\ &= A\cos\big[(k_o x - \omega_o t) + (\Delta k\, x - \Delta\omega t)\big] + A\cos\big[(k_o x - \omega_o t) - (\Delta k\, x - \Delta\omega t)\big] \\ &= A\cos(k_o x - \omega_o t)\cos(\Delta k\, x - \Delta\omega t) - A\sin(k_o x - \omega_o t)\sin(\Delta k\, x - \Delta\omega t) \\ &\quad + A\cos(k_o x - \omega_o t)\cos(\Delta k\, x - \Delta\omega t) + A\sin(k_o x - \omega_o t)\sin(\Delta k\, x - \Delta\omega t), \end{aligned}$$

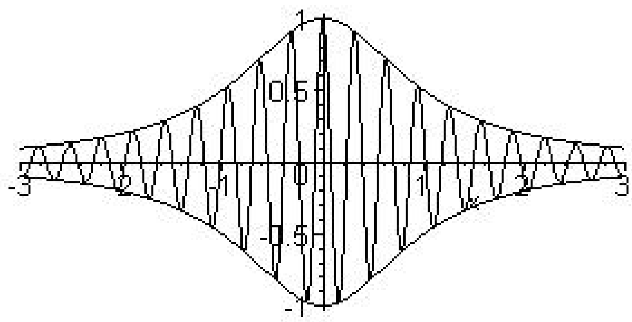
and finally

$$y = \big[2A\cos(\Delta k\, x - \Delta\omega t)\big]\cos(k_o x - \omega_o t). \tag{60}$$

This result has the same form as Eq. (56), a rapidly varying wave motion characterized by the constants k_o and ω_o, with an amplitude that varies more slowly in both space and time, as characterized by the constants Δk and $\Delta\omega$.

At time $t = 0$ the appearance of this wave looks just like the graph of y as a function of *time* (on page 13-14), but now we are plotting a graph of y as a function of x, i.e., the *shape* of the string, at time $t = 0$. If the speed of propagation c is the same for both waves in Eq. (58), the entire pattern represented by Eq. (60) moves in the $+x$ direction with constant speed c. One might well imagine it as resembling a string of short, fat sausage links moving along the x axis with constant speed c.

It is worth noting that superposing the two sinusoidal waves has the effect of concentrating the wave disturbance in certain regions along the rope (the sausages), and decreasing it in other regions (the pinched places between the sausages). We could create an even more localized disturbance by adding two more sinusoidal waves to cancel out alternate sausages in the string. We can even superpose an *infinite* set of sinusoidal waves, centered around some angular frequency ω_o and wave number k_o, using a formulation known as a Fourier integral.



Thus by superposing many sinusoidal waves we can construct a wave that is in a sense *localized* in space (in contrast to the individual sinusoidal waves, which have no end). Such a wave is called a *wave packet* or a *wave pulse*. This construction is of central importance in quantum mechanics; it helps us to understand how what we call a *particle* can have both particle and wave properties at the same time.

Now we return to the question of what happens if the wave speed c is different for different frequencies. It's a little hard to imagine this for waves on a rope, but it is a familiar phenomenon for light and other electromagnetic radiation. The refractive index of a transparent material such as glass is the ratio of the speed of light in vacuum to the speed in the material. This varies with frequency; for glass it is greater for violet light than for (lower-frequency) red light. In this case $c \ (= \omega/k)$ decreases with increasing frequency. This phenomenon is called *dispersion* The angular frequency ω is no longer proportional to the wave number k, but increases more slowly than k.

In Eq. (60), the speed of propagation of the rapid sinusoidal oscillations is

$$c_o = \frac{\omega_o}{k_o}, \tag{61}$$

while the speed of propagation of the envelope curve is

$$c_{env} = \frac{\Delta\omega}{\Delta k}. \tag{62}$$

In the case of glass, discussed above, where ω increases less than proportionately with k, $c_{env} < c_o$. The envelope curves travel at constant speed c_{env}, while the rapidly-varying oscillations inside the envelope move with a greater speed c_o, appearing at the left side of the envelope and moving out the right side. This is hard to describe, but a simple Maple demonstration helps to clarify it.

The speed c_{env} of the envelope (and thus of a wave pulse such as was described above) is called the *group velocity*, and the speed c_o of the central-frequency sinusoidal wave is called the *phase velocity*. This distinction is crucial in many areas of physics. A sinusoidal wave, having no beginning or end, can't convey information from one point to another; the maximum speed of transmission of information is the group velocity. There are situations where the phase velocity of a wave is greater than the speed of light in vacuum. This might seem to violate a basic principle of relativity, but in all such cases the *group* velocity is less than the speed of light, and so there is no violation. Finally, we note that if there is *no* dispersion, then the phase and group velocities are equal.