6 Phase Plots and Vector Field Plots Fall 2003

Phase Plots

When a particle moves along the x axis, we often represent the motion as a graph of the coordinate x or the velocity $v = dx/dt$ as a function of time t. A different but very useful representation is to plot the instantaneous position x and velocity v of the particle as a point in a plane called the *phase plane*, with horizontal and vertical axes representing $x(t)$ and $v(t)$, respectively. Such a plot is called a *phase plot*. Each point in the x-v phase plane represents an instantaneous state of motion (position and velocity) of the system. As the motion progresses, the representative point (the *phase point*) traces out a path called the *phase trajectory* in the phase plane.

A simple example is the phase plot for the undamped, undriven harmonic oscillator. The total energy E of the system is constant, and conservation of energy gives the equation

$$\tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 = E = \text{constant}. \qquad (1)$$

The graph of this equation in the x-v plane (i.e., the phase plot) is an *ellipse*. As the motion evolves in time, the phase point moves around this ellipse, tracing out the phase plot once each cycle. It always moves in the *clockwise* sense; can you see why? For *every* periodic motion, the phase plot is a closed curve that is traced out once each cycle.

When damping is present, the motion is not strictly periodic. The phase trajectory is no longer a closed curve but a spiral that curves into the origin as the motion dies down.

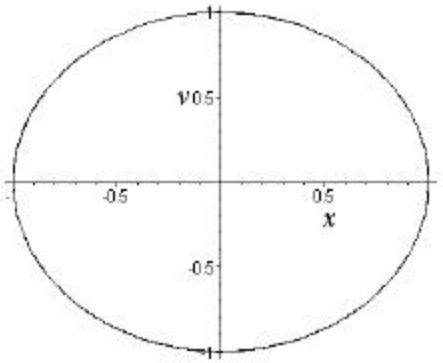
If the equation of motion (from $\Sigma F = ma$) can be solved exactly, it is easy to plot the phase trajectory. For the undamped harmonic oscillator, one solution is

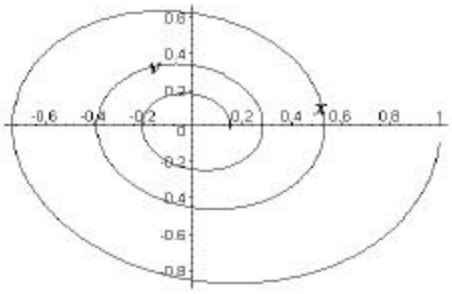
$$x = A\cos\omega_o t\,. \qquad (2)$$

For an underdamped oscillator, one solution is

$$x = Ae^{-\gamma t}\cos\omega_d t. \qquad (3)$$

Taking $A = 1$, $\omega_o = 1$, and $\gamma = 0.1$, we obtain the phase plots shown below.





For a system in which the total force depends only on x and v, that is, $F(x,v)$, knowing a single point (x_o, v_o) on the phase trajectory is always sufficient to determine the entire trajectory, and hence the entire motion of the particle. To see why this is so, we note that the equation of motion obtained from Newton's second law ($\Sigma F = ma$), can always be represented in terms of two coupled first-order equations in x and v, namely,

$$\frac{dx}{dt} = v \qquad \text{and} \qquad \frac{dv}{dt} = \frac{1}{m} F(x, v) \tag{4}$$

If the values of x and v at one instant are known, the *change* in each quantity during the following small time interval Δt can be computed:

$$\Delta x = \frac{dx}{dt} \Delta t = v \, \Delta t \qquad \text{and} \qquad \Delta v = \frac{dv}{dt} \Delta t = \frac{F(x, y)}{m} \Delta t \, . \tag{5}$$

In this way the position of a neighboring point on the phase trajectory can be computed. By iterating this process, we can construct the entire trajectory. By making Δt very small, we can compute the trajectory with as great precision as we like (with some significant exceptions). In fact, this is the basic idea behind numerical solution of differential equations, using Maple or similar systems.

The phase trajectory is a convenient way to represent general features of the motion. The possible phase trajectories can have various shapes. If the trajectory is a closed curve, as with the undamped oscillator, the resulting motion is *periodic*. The trajectory of an underdamped oscillator spirals in asymptotically (at large t) toward the fixed point ($x = 0$, $v = 0$), independently of initial conditions. Such a point is called an *attractor*.

A system can have more than one attractor; an example is a particle in a two-well potential-energy function, with two attractors. Which attractor represents the final state depends on initial conditions, and the choice can be very sensitive to small changes in initial conditions. Each attractor corresponds to a range of initial conditions, a region in phase space that is called a *basin of attraction*.

The attractor need not be a point or a set of points. Suppose the damping force is given by $F = av - bv^3$, where a and b are positive constants. For large v, F is opposite to v and is a damping force. But when v is small, F has the *same* direction as v and tends to build up the oscillations. In this case the motion approaches a periodic motion that is independent of initial conditions and in which, on the average over a cycle, the two damping terms cancel out. This final steady-state oscillation is called a *limit cycle* or *limit-cycle attractor*.

Vector Field Plots

At any point in the phase plane, the *direction* of a trajectory passing through that point is determined by Eqs. (5). We can think of Δx and Δv as the components of a vector in the x-

v phase plane. At each point this vector must be *tangent* to the trajectory passing through the point. Usually, in order to avoid undue clutter on the phase plane, we plot a *unit* vector at each point in a grid of representative points, showing the direction of a phase trajectory passing through each point. This representation is called a *vector field plot*, or simply a *field plot.*

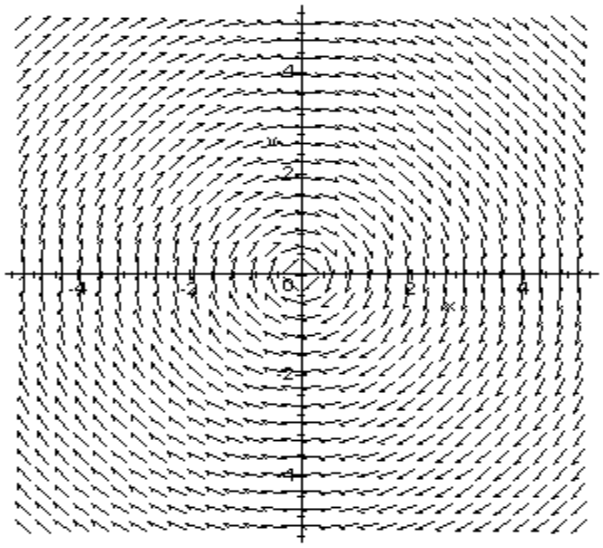
A simple example is the field plot for undamped simple harmonic motion. Equations (5) take the form

$$\Delta x = v\,\Delta t, \qquad \Delta v = -\frac{k}{m}\,x\,\Delta t \tag{6}$$

For simplicity, we take $k = 1$, $m = 1$; then

$$\Delta x = v\,\Delta t, \qquad \Delta v = -x\,\Delta t\,. \tag{7}$$

Here is a vector field plot (made with Maple) for this simple motion.



An important feature of vector field plots is that they can be obtained without actually *solving* the differential equations. This is particularly useful when solutions cannot be expressed in closed form but have to be obtained by numerical approximation. The vector field plot can always be obtained from Eqs. (5) and used to understand general features of the motion.

To compute a unit vector on a vector field plot, we introduce the unit vectors $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ in the horizontal and vertical directions, respectively, in the phase plane. Then, from Eqs. (5), the vector that represents the displacement of the representative point in the phase plane during time Δt is

$$\hat{\mathbf{i}}\,\Delta x + \hat{\mathbf{j}}\,\Delta v = \hat{\mathbf{i}}\,v\,\Delta t + \hat{\mathbf{j}}\,\frac{F(x,v)}{m}\,\Delta t. \tag{8}$$

To obtain a unit vector $\hat{\mathbf{n}}$ in the direction of this vector, we divide it by its magnitude, which is the square root of the sum of squares of components. From Eqs. (5),

$$\left|\hat{\mathbf{i}}\,\Delta x + \hat{\mathbf{j}}\,\Delta v\right| = \sqrt{(v\,\Delta t)^2 + \left[F(x,v)\,\Delta t/m\right]^2}\,.$$

Thus the unit vector $\hat{\mathbf{n}}$ at each point (x, v) in the phase plane is given by

$$\hat{\mathbf{n}} = \frac{\hat{\mathbf{i}}\,v + \hat{\mathbf{j}}\,F(x,v)/m}{\sqrt{v^2 + \left[F(x,v)/m\right]^2}}. \tag{9}$$

Using this, we can compute the components of $\hat{\mathbf{n}}$ for any point (x, v) in the phase plane, and hence construct the field plot. Note again that we have not *solved* the differential equations for $x(t)$ and $v(t)$; all that is needed to construct the field plot is the function $F(x, v)$.

Once we have the field plot, we can sketch out a phase trajectory (representing a possible motion of the system) for any initial conditions (x_o, v_o), represented by the starting point in the phase plane. We just start at (x_o, v_o) and sketch a curve that is tangent to the unit vector $\hat{\mathbf{n}}$ at each point. In the above example, it is easy to see that the phase trajectories are concentric circles.

Note that a phase trajectory can never cross itself. At each point (x, v) in the phase plane, the direction of the unit vector $\hat{\mathbf{n}}$ is determined by the values of x and v, according to Eq. (9).

Time-Dependent Forces

The total force acting on the particle may depend on time; then we denote it as $F(x, v, t)$. A familiar example is the damped, sinusoidally driven harmonic oscillator, for which

$$F(x,v,t) = -kx - bv - F_o \cos \omega t. \tag{10}$$

The spring force depends on x, the damping force depends on v, and the driving force depends on t. In this case, Eq. (9) must be rewritten as

$$\hat{\mathbf{n}} = \frac{\hat{\mathbf{i}}\,v + \hat{\mathbf{j}}\,F(x,v,t)/m}{\sqrt{v^2 + \left[F(x,v,t)/m\right]^2}}. \tag{11}$$

The unit vector $\hat{\mathbf{n}}$ at each point in the phase plane now varies with time, and a phase trajectory *can* cross itself. Suppose that at a certain time the phase point is at (x_1,v_1), and then it returns to this same point at a later time; x and v are the same as before, but t is different, and the unit vector given by Eq. (11) may have a different direction. If so, the phase trajectory crosses itself at this point. Later we will see examples of driven oscillations where the driving force is periodic but the resulting motion is chaotic, with a phase plot that is a tangled phase trajectory with many crossings.

When a periodic driving force is present, the forced-oscillation motion of the system may, in the limit of large t, approach a periodic motion that is independent of initial conditions. Such a limit is represented as a closed curve on the phase plot, and it is again called a *limit cycle.* The

phase plot may be a curve that spirals in or out, asymptotically approaching the limit-cycle curve. A limit cycle is also considered to be an attractor. There may be more than one possible limit cycle, as with the two-well potential. In such a case, initial conditions determine which limit cycle represents the final state of motion, and again the choice can be very sensitive to small changes in initial conditions.

In some cases where there is a periodic driving force, the resulting forced oscillation is *not* periodic. In such cases, the phase trajectory does not approach a limit cycle, but wanders around the phase plane. The details of the trajectory may depend very critically on the initial conditions, and predicting the long-term motion of the system is impossible. Such motion is said to be *chaotic*. The study of chaos is a topic of great current interest, with applications in many areas of physics, such as turbulent flow, phase transitions, and others.

Phase Plots with Maple

For systems where the differential equation (from $\Sigma F = ma$) can be solved exactly, it is easy to make phase plots using the parametric form of the Maple `plot` command. Here's an example. The differential equation for the damped harmonic oscillator is

$$\ddot{x} + 2\gamma\dot{x} + \omega_o{}^2 x = 0. \tag{12}$$

If the system is underdamped, then one solution is

$$x = Ae^{-\gamma t}\cos\omega_d t. \tag{13}$$

As with any Maple plot, we need to substitute specific numerical values for the parameters. Suppose we choose $A = 1$, $\omega_d = 1$, and $\gamma = 0.1$. You can verify that this corresponds approximately to the initial conditions (at time $t = 0$) $x_o = 1$, $v_o = -0.1$. To plot the phase plot for the interval $t = 0$ to 6π, corresponding to three cycles of the damped oscillation, a possible Maple scheme would be

```
restart;
x := exp(–0.1*t)*cos(t);
v := diff(x, t);
plot([x, v, t = 0..6*Pi]);
```

If the differential equation has to be solved *numerically*, then we have to use `odeplot`. Suppose we choose to solve Eq. (12) numerically, using the same numerical values as above. The Maple code goes like this:

```
restart;
with(plots, odeplot);
diffeq := diff(x(t), t$2) + 0.2*diff(x(t), t) + x(t) = 0;
init1 := x(0) = 1;
init2 := D(x)(0) = -0.1;
```

```
solution := dsolve({diffeq, init1, init2}, x(t), numeric);
odeplot(solution, [x(t), diff(x(t), t)], 0..6*Pi);
```

The range of values of t is *not* inside the square brackets, and it is stated as `0..6*Pi`, *not* `t = 0..6*Pi`. This is because Maple treats `solution` as a function, not an expression.
If you're using an older version of Maple, this plot may look a little lumpy. To smooth it out, plot more points than the default number (usually 50) by adding the statement `numpoints = 100` to the `odeplot` command:

```
odeplot(solution, [x(t), diff(x(t), t)], 0..6*Pi, numpoints = 100);
```

The newer versions of Maple do this automatically.

Vector Field Plots with Maple

Maple has a very useful and easy facility for creating vector field plots. It is called `dfieldplot`. It is part of a package called `DEtools`, and (like some of the facilities in the `plots` package) it has to be loaded explicitly by using

```
with(DEtools, dfieldplot);
```

Here's how it works. Suppose you have two coupled first-order equations:

$$\frac{dx}{dt} = f(x, v), \qquad \frac{dv}{dt} = g(x, v), \tag{14}$$

where $f(x, v)$ and $g(x, v)$ are known functions. First give the equations names, such as

```
eq1 := diff(x(t), t) = f(x,v);
eq2 := diff(v(t), t) = g(x,v);
```

Then the syntax for `dfieldplot` is

```
dfieldplot({eq1, eq2}, [x(t),v(t)], t = 0..5, x = a..b, v = c..d);
```
(15)

The equations are enclosed in curly brackets because they are a *set*. $x(t)$ and $v(t)$ are enclosed in square brackets because they are a *list*. (They are the coordinates in phase space, and the order designates which is on the horizontal axis and which on the vertical.) The ranges of values for x and v are shown in the usual way. Maple also insists on a range of values for t, which is irrelevant for our problem because $f(x,v)$ and $g(x,v)$ aren't functions of t. So just put in some random values such as `0..5`.

The default grid of unit vectors is 20×20. You can change that with the optional statement at the end: `dirgrid = [30, 30]` or however many points you want in the grid. You can also use the usual plot options such as `color = black` and the various title statements.

Example: The code used to create the example vector field plot above is

```
restart;
with(DEtools, dfieldplot);
```

```
eq1 := diff(x(t), t) = v(t);
eq2 := diff(v(t), t) = –x(t);
dfieldplot({eq1, eq2}, [x(t), v(t)], t = 0..2, x = –5..5, v = –5..5,
    dirgrid = [30, 30],  color = black);
```

To get a vector field plot for the damped oscillation problem discussed earlier, just add the term –0.2*v(t) to the right side of eq2. Do you see why this is the right thing to do? Note that the arrows and phase trajectories now spiral in toward the origin.

The functions *f* and *g* in Eqs. (14) may also contain time *t* explicitly. In this case the dfieldplot command, Eq. (15) must include the actual range of values of *t* of interest.

Superimposing Plots

Sometimes it's useful to superimpose two plots, e.g., a vector field plot and a phase trajectory. To do this you first create what are called *plot structures*. As an example, to make a plot structure for a field plot such as Eq. (15), choose numerical values for *a*, *b*, *c*, and *d*, and use

```
plot1 :=dfieldplot({eq1, eq2}, [x(t),v(t)], t = 0..5, x = a..b, v = c..d):
```

This computes all the data needed for the plot but doesn't actually display it. End the statement with a colon, rather than a semicolon, to prevent displaying a blizzard of data. Then to *display* the plot use display(plot1); The command display is part of the plots package, so it first has to be loaded explicitly using with(plots, display);

The advantage of plot structures is that once you have created them you can superimpose two or more plots by asking Maple to display a *set* of plot structures, such as

```
display({plot1, plot2});
```

Example: The following code will produce the field plot shown on p. 6-3 (with red arrows), superimposed on a phase plot (in black) for undamped simple harmonic motion with ω = 1 and amplitude 3. Try it!

```
restart;
with(DEtools, dfieldplot);
with(plots, display);
eq1 := diff(x(t), t) = v(t);
eq2 := diff(v(t), t) = -x(t);
plot1 := dfieldplot({eq1, eq2}, [x(t), v(t)], t = 0..2, x = -5..5, v = -5..5,
```

```
    dirgrid = [30,30], color = red):
plot2 := plot([3*sin(t), 3*cos(t), t = 0..2*Pi], color = black):
display({plot1, plot2});
```