33-231 Physical Analysis Fall 2003

Problem Solutions: Set 9 (October 29, 2003)

35. a) Use force expressions from Problem 34, with additional driving force:

$$-kx + \frac{3kx^2}{a} - \frac{2kx^3}{a^2} - b\frac{dx}{dt} + F_o \cos\omega t = m\frac{d^2x}{dt^2}.$$

With suggested numerical values ($m = 1$, $k = 1$, $a = 1$, $b = 0.1$),

$$\frac{d^2x}{dt^2} + 0.1\frac{dx}{dt} + x - 3x^2 + 2x^3 = F_o \cos\omega t.$$

b) The limit cycle is more or less elliptical, with $x_{max} \cong 0.022$ and $v_{max} \cong 0.004$.

c) The limit cycle passes approximately through the points ($x = 0.022$, $v = 0$) and ($x = 0$, $v = 0.004$). If we use the first of these as initial conditions, the limit cycle is approached very quickly. Not so for the other point, because the *phase* of the driving force isn't right. But if we use the second point with a driving force $F = 0.02 \sin(0.2t)$ instead of $\cos(0.2t)$, the limit cycle is again approached quickly.

d) Chaotic motion develops when F_o is greater than about 0.097.

e) Many possibilities; for example, with $F_o = 0.1$, running to $t = 100$, phase trajectory for initial conditions ($x = 0$, $v = 0$) is very different from trajectory for ($x = 0.01$, $v = 0$).

36. a) For $0 \le x \le 1$, the maximum value of $x(1-x)$ (at $x = 1/2$) is 1/4. So if

$$a \le 4, \quad \text{then} \quad ax(1-x) \le 1 \quad \text{for} \quad 0 \le x \le 1.$$

b) We need $x = ax(1-x)$. Solve for x: $\quad x = 0, \quad x = 1 - \dfrac{1}{a}$.

(Note that when $a \le 1$, the only root in the interval $0 \le x \le 1$ is $x = 0$.)

c) Numerical experiments

d) $$x_{n+1} = a x_n (1 - x_n); \qquad x_{n+2} = a x_{n+1}(1 - x_{n+1});$$

$$x_{n+2} = a\big[a x_n (1 - x_n)\big]\big[1 - a x_n (1 - x_n)\big].$$

If $x_{n+2} = x_n$, then x must satisfy the equation

$$x = a\big[a x(1-x)\big]\big[1 - a x(1-x)\big].$$

This is a fourth degree equation. Two of the roots are already known: $x = 0$ and $x = 1 - 1/a = 0.677419$. So in principle the problem could be reduced to solving a quadratic equation. Instead, use Maple to find all four roots, which are

0, 0.558013, 0.677419, 0.764568.

The second and fourth are attractors, the other two are repellers.

e) Numerical experiments. The attractors are independent of the value of x_0.

f) The second bifurcation occurs at $a = 3.449490$.

g) When $a = 3$, $\quad y = 3x(1-x), \quad \dfrac{dy}{dx} = 3 - 6x.$

From (b), intersection point is at $x = 1 - \dfrac{1}{a} = 1 - \dfrac{1}{3} = \dfrac{2}{3}$.

Slope of parabola at this point is the value of $\dfrac{dy}{dx}$ at this point: $3 - 6\left(\dfrac{2}{3}\right) = -1.$

Slope of line is +1; so the two are perpendicular at the intersection point.

37. a) $0 \le a \le 1$.

b) $x = 0.6\sin(\pi x)$ eq := x = 0.6*sin(Pi*x); attractor := fsolve(eq, x = 0.1..1);

$x = 0$, 0.580781. (The command fsolve(eq, x); returns only the value $x = 0$.)

c) The value $x = 0.5$ satisfies the equation $x = 0.5\sin(0.5\pi)$.

In (b) and (c), use numerical experiments to show that the values of x are attractors and not repellers.

d) The first three bifurcations occur at approximately

$a = 0.719$, 0.833, 0.858

38. a)

$$-kx_1 + k(x_2 - x_1) = 3m\ddot{x}_1,$$
$$-k(x_2 - x_1) = 2m\ddot{x}_2.$$

b) Try a solution in the form $x_1 = a_1 \cos\omega t$, $x_2 = a_2 \cos\omega t$; let $\lambda = \omega^2$. Substitute, divide out the common factor $\cos\omega t$:

$$\begin{aligned} -2ka_1 + ka_2 &= -3m\omega^2 a_1, \\ ka_1 - ka_2 &= -2m\omega^2 a_2, \end{aligned} \qquad \text{or} \qquad \begin{aligned} (2k - 3m\lambda)\, a_1 - ka_2 &= 0, \\ -ka_1 + (k - 2m\lambda)a_2 &= 0. \end{aligned}$$

For non-trivial solutions, secular determinant must be zero:

$$\begin{vmatrix} 2k - 3m\lambda & -k \\ -k & k - 2m\lambda \end{vmatrix} = 0, \qquad (2k - 3m\lambda)(k - 2m\lambda) - k^2 = 0,$$

$$(6m\lambda - k)(m\lambda - k) = 0, \qquad \lambda_1 = \frac{1}{6}\frac{k}{m}, \qquad \lambda_2 = \frac{k}{m}.$$

Amplitudes: For λ_1, $(2k - \frac{1}{2}k)a_1 - ka_2 = 0$, $a_2 = \frac{3}{2}a_1$.
For λ_2, $(2k - 3k)a_1 - ka_2 = 0$, $a_2 = -a_1$.

c)

$$\left.\begin{aligned} x_1 &= a\cos\omega_1 t + b\cos\omega_2 t = q_1 + q_2, \\ x_2 &= \tfrac{3}{2}a\cos\omega_1 t - b\cos\omega_2 t = \tfrac{3}{2}q_1 - q_2 \end{aligned}\right\} \quad \text{general solution}$$

Solve for q_1 and q_2: $q_1 = \frac{2}{5}(x_1 + x_2)$, $q_2 = \frac{3}{5}x_1 - \frac{2}{5}x_2$.

(continued)

38. (continued)

d) Substitute into $\Sigma F = ma$ equations:

$$-2k(q_1 + q_2) + k(\tfrac{3}{2}q_1 - q_2) = 3m(\ddot{q}_1 + \ddot{q}_2),$$
$$-k(\tfrac{3}{2}q_1 - q_2) + k(q_1 + q_2) = 2m(\tfrac{3}{2}\ddot{q}_1 - \ddot{q}_2).$$

Subtract second equation from first:

$$-5kq_2 = 5m\ddot{q}_2, \quad \text{or} \quad \ddot{q}_2 = -\frac{k}{m}q_2, \quad \lambda_2 = \omega_2^{\,2} = \frac{k}{m}.$$

Multiply first equation by 2, second by 3, and add:

$$-\tfrac{5}{2}kq_1 = 15m\ddot{q}_1, \quad \text{or} \quad \ddot{q}_1 = -\frac{k}{6m}q_1, \quad \lambda_1 = \omega_1^{\,2} = \frac{k}{6m}.$$

e) $$E = \tfrac{3}{2}m\dot{x}_1^{\,2} + m\dot{x}_2^{\,2} + \tfrac{1}{2}kx_1^{\,2} + \tfrac{1}{2}k(x_2 - x_1)^2.$$

Substitute normal-coordinate transformation from (c) and simplify, to obtain

$$E = \tfrac{15}{4}m\dot{q}_1^{\,2} + \tfrac{5}{2}m\dot{q}_2^{\,2} + \tfrac{5}{8}kq_1^{\,2} + \tfrac{5}{2}kq_2^{\,2}.$$

The first and third terms contain only q_1, the other two contain only q_2.

f) $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{3}{2} & -1 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}, \qquad \text{so} \qquad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ \frac{3}{2} & -1 \end{pmatrix}.$$

Also, from (c),

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \text{so} \qquad \mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}.$$

Check: $$\mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{3}{2} & -1 \end{pmatrix}\begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$