33-231 **Physical Analysis** **Fall 2003**

Problem Solutions: Set 6 (October 8, 2003)

25. a) $T = 2\pi\sqrt{\frac{L}{g}}$, $g = \frac{4\pi^2 L}{T^2} = \frac{4\pi^2(1.000\text{ m})}{(2.000\text{ s})^2} = 9.87\text{ m/s}^2$.

b) To get total energy, find potential energy at points of maximum displacement:

$$V = mgh = mgL(1-\cos\theta) = (0.200\text{ kg})(9.87\text{ m/s}^2)(1.000\text{ m})\left[1 - \cos(0.100\text{ rad})\right]$$
$$= 0.00986\text{ J.}$$

Energy loss in 24 hr (= 86,400 s = 43,200 cycles) is

$mgh = (0.500\text{ kg})(9.87\text{ m/s}^2)(0.800\text{ m}) = 3.95\text{ J}$.

Energy loss *per cycle*: $|\Delta E| = (3.95\text{ J})/(43{,}200\text{ cycles}) = 9.14\times 10^{-5}\text{ J}$.

$$Q = \left|\frac{E}{\Delta E/2\pi}\right| = 2\pi\frac{0.00986\text{ J}}{9.14\times 10^{-5}\text{ J}} = 678.$$

c) A 5-J battery would power the clock for $\frac{5\text{ J}}{3.95\text{ J}}(86{,}400\text{ s}) = 1.09\times 10^5\text{ s}$,
or about 30 hours. (Note: An ordinary "D" flashlight cell has an energy capacity of the order of 10^4 J. So a battery with total energy of 5 J is likely to be a small watch or hearing-aid battery.)

26. a) General solution for critically damped case:

$$x = A'(\omega)\cos(\omega t+\varphi) + (A+Bt)e^{-\gamma t}.$$

In this problem, x_o and v_o are given, and A and B are to be determined from the initial conditions; A' and φ have their usual meaning for driven oscillations. Take derivative:

$$v = \dot{x} = -\omega A'(\omega)\sin(\omega t+\varphi) + \left[B - \gamma(A+Bt)\right]e^{-\gamma t}.$$

Substitute $x = x_o$, $v = v_o$ at time $t = 0$; solve resulting equations for A and B:

$$A = x_o - A'\cos\varphi, \qquad B = \gamma x_o + v_o + A'(\omega\sin\varphi - \gamma\cos\varphi).$$

(Remember that A' and φ depend on driving frequency ω and force amplitude F_o.)

b) Use results from part (a). We want to find values of x_o and v_o so A and B will be zero. From above equations,

$$x_o = A'\cos\varphi, \qquad v_o = -\omega A'\sin\varphi.$$

27. $f = 440$ Hz, $\omega = 2\pi(440\ \text{s}^{-1})$; $\gamma = \dfrac{\omega_o}{2Q} = \dfrac{2\pi(440\ \text{s}^{-1})}{(2)(4400)} = \dfrac{\pi}{10}\ \text{s}^{-1}$

Use light-damping approximation: $A'(\omega) = \dfrac{F_o / 2m\omega_o}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}}$

a) When $\omega = \omega_o$, $A' = \dfrac{F_o}{2m\omega_o \gamma}$;

$$F_o = 2m\omega_o \gamma A' = 2(0.010\ \text{kg})(2\pi \times 440\ \text{s}^{-1})\left(\frac{\pi}{10}\ \text{s}^{-1}\right)(0.00200\ \text{m})$$

$$= 0.0347\ \text{kg m/s}^2 = 0.0347\ \text{N}.$$

$\omega - \omega_o = 2\pi\ \text{s}^{-1}$;

b) $$\frac{A'_{441}}{A'_{440}} = \frac{\sqrt{0^2 + \gamma^2}}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}} = \frac{\pi/10}{\sqrt{(2\pi)^2 + (\pi/10)^2}} = \frac{1}{\sqrt{401}}$$

So $A'_{441} = \dfrac{1}{\sqrt{401}} A'_{440} = (0.0499)(0.00200\ \text{m}) \cong 0.10$ mm.

That is, a frequency change of 1 Hz causes a drop in amplitude of about a factor of 20.

c) Now $\gamma = \pi\ \text{s}^{-1}$. F_o must be larger by a factor of 10; $F_o = 0.347$ N.

$$\frac{A'_{441}}{A'_{440}} = \frac{\pi}{\sqrt{(2\pi)^2 + (\pi)^2}} = \frac{1}{\sqrt{5}}, \quad \text{so} \quad A'_{441}\ \frac{1}{\sqrt{5}}(2.00\ \text{mm}) = 0.89\ \text{mm}.$$

In this case a frequency change of 1 Hz causes a much *smaller* decrease in amplitude than before, and the force needed is 10 times as great for the same amplitude.

d) The angular frequency of free oscillations is $\omega_d = \sqrt{\omega_o^2 - \gamma^2}$. When Q = 4400, ω_d is very nearly equal to ω_o. Also, for any value of Q,

$$Q = \frac{\omega_o}{2\gamma}, \quad \text{so} \quad \gamma = \frac{\omega_o}{2Q} \quad \text{and} \quad \omega_d = \sqrt{\omega_o^2 - \left(\frac{\omega_o}{2Q}\right)^2} = \omega_o \sqrt{1 - \frac{1}{4Q^2}}.$$

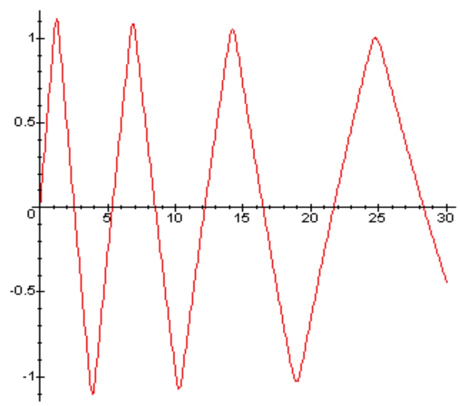
For a frequency difference of 0.10 Hz,

$$\frac{\omega_d}{\omega_o} = \frac{439.9\ \text{Hz}}{440\ \text{Hz}} = \sqrt{1 - \frac{1}{4Q^2}} \quad \text{and} \quad Q \cong 23.4$$

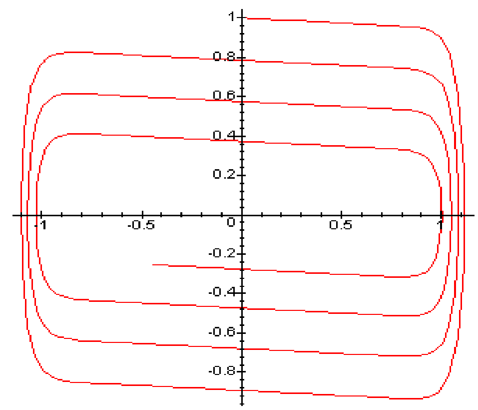
28. a) and b)   `diffeq := diff(x(t), t$2)  +  (0.1)*diff(x(t),t) + x(t)^19 = 0;`

The following plots were made with $x_o = 0$, $v_o = 1$, and $b = 0.05$.

$x(t)$



v vs. x



The amplitude is nearly constant for several cycles, but the period grows as the particle slows down. The straight segments on the phase plot are regions where the particle's velocity is nearly constant because the force (and acceleration) are very small. When b is larger, the particle may almost stop at some area in the middle.

29. Yes, x^{99} is better; the force is then a better approximation to a rigid wall. The force is essential zero until x reaches ± 1;   then it suddenly becomes very large.

A possible Maple differential equation is

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diffeq := diff(x(t), t$2)  +  (0.1)*signum(diff(x(t), t))*diff(x(t), t)^2
    +  x(t)^99  =  0;
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Some possibilities: plot $x(t)$, (observe changing period), phase plot, comparison with linear velocity dependence of drag force in Prob. 28.