33-231 Physical Analysis Fall 2003

Problem Solutions: Set 5 (October 1, 2003)

20. a) $T = 2\pi\sqrt{\dfrac{L}{g}} \; = \; 2\pi\sqrt{\dfrac{1.00\,\text{m}}{9.80\,\text{m/s}^2}} \; = \; 2.01\ \text{s.}$

$$f = \frac{1}{2\pi}\sqrt{\frac{g}{L}} \; = \; \frac{1}{2\pi}\sqrt{\frac{9.80\ \text{m/s}^2}{1.00\,\text{m}}} \; = \; 0.498\ \text{Hz}, \qquad \omega_o \; = \; \sqrt{\frac{g}{L}} \; = \; 3.13\,\text{s}^{-1}.$$

b) $mgh = \tfrac{1}{2}mv_o^{\,2}, \quad mg(2L) = \tfrac{1}{2}mv_o^{\,2},$

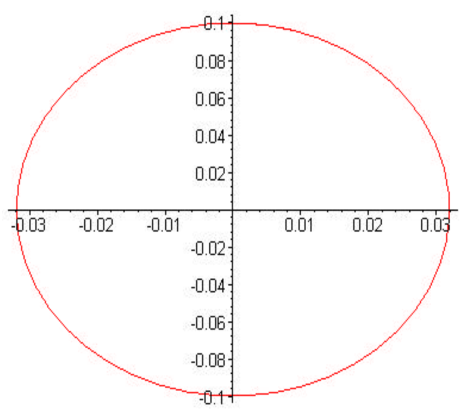
$$v_o \; = \; \sqrt{4gL} \; = \; \sqrt{4(9.80\ \text{m/s}^2)(1.00\ \text{m})} \; = \; 6.26\ \text{m/s}\,.$$

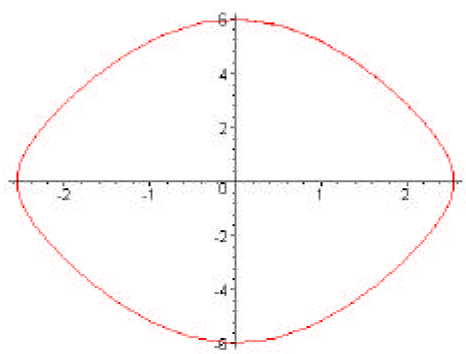
c) $T = 2.01$ s.

d) When $v_o = 3.6$ m/s, $T = 2.21$ s.

e) When $v_o > \sqrt{4gL}$, pendulum goes all the way around and x increases continuously.

f) Phase plot for $v_o = 0.100$ m/s is an ellipse. For $v_o = 6.00$ m/s it is more football-shaped.



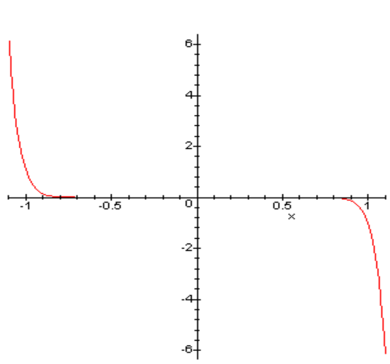


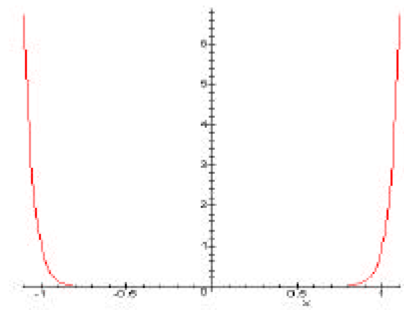
21. a) Curve is nearly flat when $|x| < a$, but it rises steeply near $x = -a$ and drops steeply near $x = a$.

b) $$V(x) = \frac{ka}{20}\left(\frac{x}{a}\right)^{20} \quad (+\ \text{constant}).$$

c) $[m]$ = kg, $[a]$ = meters, $[k]$ = newtons or kg m/s^2.

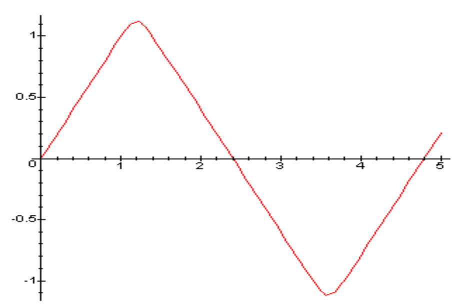
d) $F(x)$ $V(x)$





e)

```
with(plots, odeplot);
diffeq := diff(x(t), t$2) = –x^19;
init1 := x(0) = 0;     init2 := D(x)(0) = 1;
sol := dsolve({diffeq, init1, init2}, x(t), numeric);
odeplot(sol, 0..5, numpoints = 100)
```

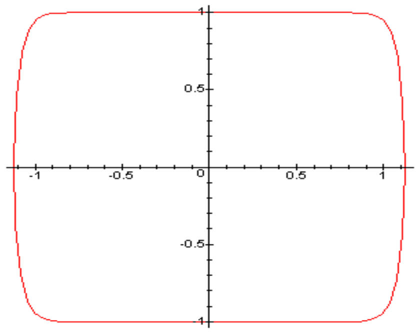


Period is about 4.8. With $v_o = 2$, period is about 2.6. Period is approximately inversely proportional to v_o.

(continued)

21. (continued)

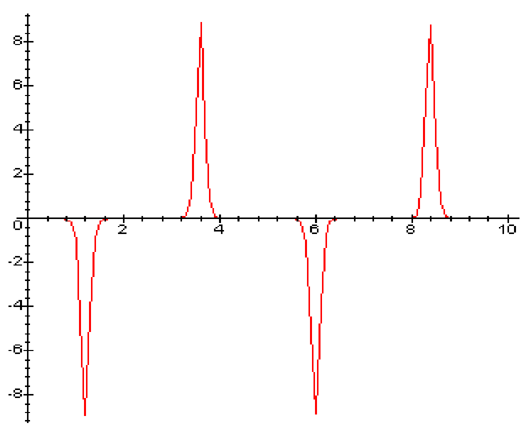
f) `odeplot(sol, [x(t), diff(x(t), t)], 0..5, numpoints = 100);`



Phase plot is nearly rectangular. Particle moves with nearly constant velocity until it runs into steep parts of $F(x)$.

g) The distance between the walls is $2a$, so the particle must travel a distance of $4a$ during one period (T). Thus $4a = v_o T$ or $T = \dfrac{4a}{v_o}$. In part (e), with $a = 1$, when $v_o = 1$, $T = 4.8$; the equation predicts 4.0. When $v_o = 2$, $T = 2.6$; the equation predicts 2.0. Thus there is fair agreement.

h) `odeplot(sol, [t, -x(t)^19], 0..10, numpoints = 100);` (Extend range to show two cycles.)



22. a) For critical damping, $\omega_o = \gamma$, and the general solution is $x = (A + Bt)e^{-\gamma t}$.
At time $t = 0$, this becomes $x_o = A$.
Also, $\dot{x} = v = Be^{-\gamma t} - \gamma(A + Bt)e^{-\gamma t}$. At time $t = 0$, this becomes
$v_o = B - \gamma A = B - \gamma x_o$. So the constants A and B are $A = x_o$, $B = v_o + \gamma x_o$.

b) For overdamping, $\gamma > \omega_o$. Let $\gamma_d = \sqrt{\gamma^2 - \omega_o{}^2}$. Then the general solution is

$$x = e^{-\gamma t}\left(Ae^{\gamma_d t} + Be^{-\gamma_d t}\right) = Ae^{(\gamma_d - \gamma)t} + Be^{-(\gamma + \gamma_d)t}.$$

At time $t = 0$ this becomes $x_o = A + B$. Also,

$$\dot{x} = (\gamma_d - \gamma)Ae^{(\gamma_d - \gamma)t} - (\gamma + \gamma_d)Be^{-(\gamma + \gamma_d)t}.$$ At time $t = 0$ this becomes

$$v_o = (\gamma_d - \gamma)A - (\gamma + \gamma_d)B.$$ Solve these two equations for A and B.

One procedure is to solve the first for B, then substitute this into the second to eliminate B, then solve for A. Results:

$$A = \frac{v_o + (\gamma + \gamma_d)x_o}{2\gamma_d}, \qquad B = \frac{(\gamma_d - \gamma)x_o - v_o}{2\gamma_d}.$$

For each part of the problem, substitute the expressions for A and B into the general solutions to obtain a complete expression for $x(t)$ in terms of the initial conditions.

23. To find times when maxima and minima occur, take dx/dt and set it equal to zero:

$$\dot{x} = Ae^{-\gamma t}(-\gamma \cos\omega_d t - \omega_d \sin\omega_d t).$$ This is zero when $$\tan\omega_d t = -\frac{\gamma}{\omega_d}.$$

This equation has infinitely many roots, since $\tan(\omega_d t + n\pi) = \tan\omega_d t$ (where n is any integer). So successive maxima and minima are separated by a time Δt such that $\omega_d \Delta t = \pi$, and successive *maxima* are separated by a time interval such that $\omega_d \Delta t = 2\pi$, or $\Delta t = 2\pi/\omega_d$. Thus the factor $\cos\omega_d$ has the same value at all maxima, and the ratio of the values of x at two successive maxima is just the ratio of the values of the exponential factor, namely

$$\frac{e^{-\gamma(t + 2\pi/\omega_d)}}{e^{-\gamma t}} = e^{-2\pi\gamma/\omega_d}.$$

If A_1 and A_2 are the displacements at two successive maxima, then

$$\frac{A_2}{A_1} = e^{-2\pi\gamma/\omega_d}.$$

24. a) If $\omega_d = \frac{12}{13}\omega_o$, then $\sqrt{\omega_o^2 - \gamma^2} = \frac{12}{13}\omega_o$ and $\gamma = \frac{5}{13}\omega_o$.

b) $\left|\frac{\Delta A}{A_1}\right| = \frac{A_1 - A_2}{A_1} = 1 - \frac{A_2}{A_1}$. From Problem 23,

$$\left|\frac{\Delta A}{A_1}\right| = 1 - e^{-2\pi\gamma/\omega_d} = 1 - e^{-2\pi(5\omega_o/13)/(12\omega_o/13)} = 1 - e^{-5\pi/6}$$

c) At a point of maximum displacement, there is no kinetic energy, so the potential energy equals the total energy. Thus $E_1 = \frac{1}{2}kA_1^2$, etc.

$$\left|\frac{\Delta E}{E_1}\right| = \frac{E_1 - E_2}{E_1} = 1 - \frac{E_2}{E_1} = 1 - \frac{A_2^2}{A_1^2}.$$ From Problem 23,

$$\left|\frac{\Delta E}{E_1}\right| = 1 - e^{-4\pi\gamma/\omega_d} = 1 - e^{-5\pi/3}.$$

With this large amount of damping, the energy and displacement decrease to a small fraction of their previous values after only one cycle. But for a system with very little damping (e.g., $\gamma = 0.01\omega_o$),

$$\left|\frac{\Delta A}{A_1}\right| = 1 - e^{-2\pi(0.01)} = 0.061 \quad \text{and} \quad \left|\frac{\Delta E}{E_1}\right| = 1 - e^{-4\pi(0.01)} = 0.118.$$

In that case the decreases in A and E during one cycle are relatively small fractions of the values at the beginning of the cycle.