Problem Solutions: Set 4 (September 24, 2003)

15. a) $$k = \left|\frac{F}{x}\right| = \frac{0.480\text{ N}}{0.100\text{ m}} = 4.80\text{ N/m}.$$

b) $$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{4.80\text{ N/m}}{0.300\text{ kg}}} = 4.00\text{ s}^{-1}, \quad f = \frac{\omega_o}{2\pi} = 0.637\text{ Hz}, \quad T = \frac{1}{f} = 1.57\text{ s}.$$

c) $x_o = 0.0300$ m, $v_o = 0.160$ m/s.

$$A = \sqrt{x_o^{\,2} + \left(\frac{v_o}{\omega_o}\right)^2} = \sqrt{(0.0300\text{ m})^2 + \left(\frac{0.160\text{ m/s}}{4.00\text{ s}^{-1}}\right)^2} = 0.0500\text{ m}.$$

$$\varphi = \arctan\left(\frac{-v_o}{\omega_o x_o}\right) = \arctan\left(\frac{-0.160\text{ m/s}}{(4.00\text{ s}^{-1})(0.0300\text{ m})}\right) = -53.1^\circ = -0.927\text{ rad}.$$

$$\text{Energy} = \tfrac{1}{2} m v_o^{\,2} + \tfrac{1}{2} k x_o^{\,2}$$
$$= \tfrac{1}{2}(0.300\text{ kg})(0.160\text{ m/s})^2 + \tfrac{1}{2}(4.80\text{ N/m})(0.0300\text{ m})^2 = 0.00600\text{ J}.$$

d) Yes, with very little effort. The force required on each end is

F = (4.80 N/m)(0.0300 m) = 0.144 N = 0.0323 lb = about 0.5 oz..

e) $x = (0.0500\text{ m})\cos\left[(4.00\text{ s}^{-1})t - 0.927\right]$. Compare the graph of this function with the graph of $x = (0.0500\text{ m})\cos\left[(4.00\text{ s}^{-1})t\right]$.

16. a) $$V = V_o\left(\frac{a^2}{x^2} - \frac{a}{x}\right) = -\frac{V_o}{4} + \frac{V_o}{16a^2}(x-2a)^2 - \frac{V_o}{16a^3}(x-2a)^3 + \cdots ,$$

$$F = V_o\left(\frac{2a^2}{x^3} - \frac{a}{x^2}\right) = -\frac{V_o}{8a^2}(x-2a) + \frac{3V_o}{16a^3}(x-2a)^2 + \cdots .$$

Force constant $= k = \dfrac{V_o}{8a^2}$;

$$\omega_o = \sqrt{\frac{k}{m}} = \sqrt{\frac{V_o}{8ma^2}} = \sqrt{\frac{5.00\text{ J}}{8(0.500\text{ kg})(2.00\text{ m})^2}} = 0.559\text{ s}^{-1}. \quad T = \frac{2\pi}{\omega_o} = 11.2\text{ s}.$$

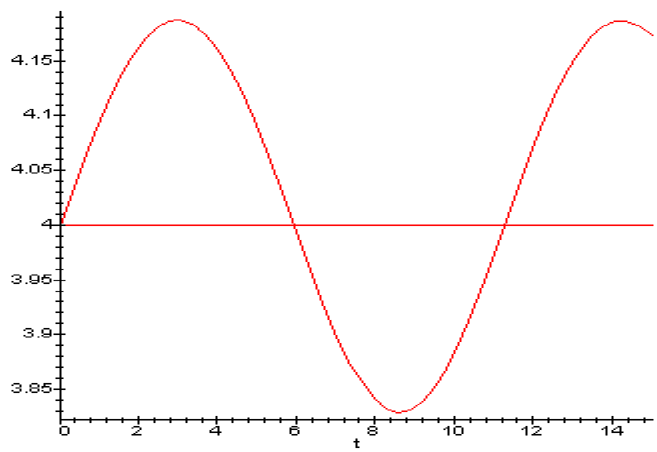
b) When the second term in the series for F is 0.10 times the first term,

$$\frac{3V_o}{16a^3}(x-2a)^2 = \pm(0.10)\frac{V_o}{8a^2}(x-2a), \quad \text{and} \quad x \cong 2a \pm 0.07a.$$

17. a) $F = V_o\left(\frac{2a^2}{x^3} - \frac{a}{x^2}\right)$; $V_o = 5.00\text{ J}$, $a = 2.00\text{ m}$, $m = 0.500\text{ kg}$,

$$F = 5\left(\frac{8}{x^3} - \frac{2}{x^2}\right) = (0.5)\frac{d^2x}{dt^2}; \qquad x_o = 4, \quad v_o = 0.1.$$

b)
```
with(plots, odeplot);        F := 5*(8/x^3  - 2/x^2);
diffeq := F = (0.5)*diff(x(t), t$2);
init1 := x(0) = 4;       init2 := D(x)(0) = 0.1;
sol :=dsolve({diffeq, init1, init2}, x(t), numeric);
odeplot(sol, [t, x(t)], 0..15);
```



Period appears to be about 11 s; the result from Problem 16(a) is 11.2 s.

c) $v_o = 0.5$, $T = 12.2$; $v_o = 1.0$, $T = 16$; $v_o = 1.5$, $T = 28$

Oscillations become more conspicuously asymmetric as v_o increases.

d) From Problem 14(e), at equilibrium $V(x) = -\frac{V_o}{4} = -1.25\text{ J}$ For the particle to escape from the well, its kinetic energy at this point must be at least 1.25 J.

Thus $\frac{1}{2}mv_{esc}^2 = \frac{V_o}{4}$, or $v_{esc} = \sqrt{\frac{V_o}{2m}} = \sqrt{\frac{5\text{ J}}{2(0.5\text{ kg})}} = 2.236\text{ m/s}$

18. a) From class discussion, $T = \sqrt{2m}\int_{x_{min}}^{x_{max}} \frac{dx}{\sqrt{E - V(x)}}$. Using $V(x)$ and numerical values from Problem 14(c,d):

```
V := 5*(4/x^2 - 2/x);
T := sqrt(2*0.5)*evalf(int(1/(-1 - V)^(1/2),  x = 2.764..7.236));
```

Maple numerical integration gives $T = 15.6$ s.

b) At equilibrium point, $V(x) = -1.25$ J and $E = -1$ J, so the kinetic energy is

$$\frac{1}{2}mv^2 = \frac{1}{2}(0.5\text{ kg})v^2 = 0.25\text{ J}, \quad \text{or} \quad v = 1 \text{ m/s}.$$

c) From Problem 17(c), when $v_o = 1$ m/s, the estimated period is 16 s; reasonably good agreement (within precision of estimate from graph).

19. a) $$V(x) = k\left[\sqrt{L^2 + x^2} - L\right]^2.$$

b) $$F(x) = -\frac{2k\left(\sqrt{L^2 + x^2} - L\right)x}{\sqrt{L^2 + x^2}}$$

c) $F(x) = -\frac{k}{L^2}x^3 + \frac{3k}{4L^4}x^5 + \cdots$. The series has only odd powers of x, as should be expected from the symmetry of the situation. The surprise is that there is no term containing x to the first power, so there is no linear approximation to F. This means that the motion is *never* simple harmonic, no matter how small the amplitude.

d) $$\frac{d^2x}{dt^2} = -x^3; \qquad x(0) = 0, \qquad x'(0) = v_o.$$

```
with(plots, odeplot);
diffeq := diff(x(t), t$2) = -x^3;
init1 := x(0) = 0;  init2 := D(x)(0) = .1;
sol :=dsolve({diffeq, init1, init2}, x(t), numeric);
odeplot(sol, [t, x(t)], 0..25);
```

When $v_o = 0.1$, $T = 20$; when $v_o = 1$, $T = 6.3$, etc. Period definitely depends on amplitude. The graph of $x(t)$ is similar to a sine curve but is more peaked. To get the phase plot, replace the `odeplot` command above with

```
odeplot(sol, [x(t), diff(x(t), t)], 0..25);
```

The phase plot is not elliptical but is somewhat pumpkin-shaped.