Physical Analysis

Problem Solutions: Set 2  (September 10, 2003)

6.  a) Units of $a$ are $m^{-1}s^{-2}$, i.e., (meters)$^{-1} \times$ (seconds)$^{-2}$.

   b) $x = \frac{2}{at^2} + \text{(constant)}$.

   d) $x = \frac{2}{4t^2 + 1}$.

   g) $t = \pm \frac{1}{2} \text{ s}$.

7.  a) From $\Sigma = ma$, $\frac{dv}{dt} = -g - bv$. Separate variables and integrate:

\[
\frac{dv}{v + \frac{mg}{b}} = - \frac{b}{m} \, dt,
\ln\left(v + \frac{mg}{b}\right) = - \frac{b}{m} \, t + \text{const.}
\]

\[
v + \frac{mg}{b} = e^{-(b/m)t + \text{const}} = \text{(const)} \, e^{-(b/m)t}.
\]

At time $t = 0$, $v_o + \frac{mg}{b} = \text{const}$. Substitute and re-arrange to obtain

\[
v = - \frac{mg}{b} + \left(v_o + \frac{mg}{b}\right) e^{-(b/m)t} = \frac{dy}{dt}.
\]

Separate variables and integrate again to obtain

\[
y = - \frac{mg}{b} \, t + \left(v_o + \frac{mg}{b}\right) \left(-\frac{m}{b}\right) e^{-(b/m)t} + \text{const.} \quad \text{At time } t = 0,
\]

\[
y_o = \left(v_o + \frac{mg}{b}\right) \left(-\frac{m}{b}\right) + \text{const} \quad \text{Substitute into previous equation to obtain}
\]

\[
y = y_o - \frac{mg}{b} \, t + \frac{m}{b} \left(v_o + \frac{mg}{b}\right) \left(1 - e^{-(b/m)t}\right).
\]

b) See bboard post for Maple code.

c) Substitute into previous results and simplify.
8. a) \( e^{-(b/m)t} = 1 - \frac{b}{m}t + \frac{b^2t^2}{2m^2} + \cdots \).

b) \( v = -\frac{mg}{b} + \left( v_o + \frac{mg}{b} \right) \left( 1 - \frac{b}{m}t + \frac{b^2t^2}{2m^2} + \cdots \right) \)

\[ = v_o - gt - \frac{b}{m}v_o t + \frac{mg}{b} \frac{b^2t^2}{2m^2} + \frac{mb^2t^2}{2bm^2} + \cdots \]  

Because \( t \) is finite, the last three terms (and all successive terms) go to zero as \( b \to 0 \), leaving \( v = v_o - gt \), as expected.

c) \( y = y_o - \frac{mg}{b}t + \frac{m}{b} \left( v_o + \frac{mg}{b} \right) \left[ 1 - \left( 1 - \frac{b}{m}t + \frac{b^2t^2}{2m^2} + \cdots \right) \right] \).

Multiply out:

\( y = y_o - \frac{mg}{b}t + v_o t + \frac{mg}{b}t - \frac{v_o t^2 b}{2m} - \frac{1}{2} gt^2 \).

The second and fourth terms cancel and the fifth (and all successive terms) go to zero as \( b \to 0 \), leaving

\( y = y_o + v_o t - \frac{1}{2} gt^2 \), as expected.

9. a) \( -bv^2 + mg = m \frac{dv}{dt} \). When \( \frac{dv}{dt} = 0 \), \( v = v_T = \sqrt{\frac{mg}{b}} \).

The units of \( b \) are \( \text{force} \ (\text{velocity})^2 = \frac{\text{kg} \ \text{m/s}^2}{\text{m}^2/\text{s}^2} = \frac{\text{kg}}{\text{m}} \). The units of \( v_T \) are then \( \sqrt{\frac{\text{kg} \ \text{m/s}^2}{\text{kg}/\text{m}}} = \text{m/s} \), as expected.

b) \( v = \sqrt{\frac{mg}{b}} \tanh \sqrt{\frac{gb}{m}} t \). Let \( \tau = \sqrt{\frac{m}{gb}} \); then \( v = v_T \tanh \frac{t}{\tau} \).

The units of \( \tau \) are \( \sqrt{\frac{\text{kg}}{(\text{m/s})^2 \text{(kg/m)}}} = \text{s} \).

(as required to make the quantity \( t/\tau \) dimensionless).

(continued)
9. (continued)

c) \[ y = \frac{m}{b} \ln \left[ \cosh \left( \sqrt{\frac{gb}{m}} t \right) \right] = v_T \tau \ln \left[ \cosh \left( \frac{t}{\tau} \right) \right]. \]

e) From Maple `taylor` command, the first term in the Taylor expansion of \( v(t) \) reduces to \( gt \). All other terms contain positive powers of \( b \) and therefore go to zero as \( b \to 0 \). Similarly, the first term in the Taylor series expansion of \( y(t) \) is \( gt^2/2 \). All other terms contain positive powers of \( b \).

10. a) \[ b = \frac{CA\rho}{2} = \frac{(0.8)(1 \text{ m}^2)(1.2 \text{ kg/m}^3)}{2} = 0.5 \text{ kg/m}. \]

\[ v_T = \sqrt{\frac{mg}{b}} = \sqrt{\frac{(80 \text{ kg})(10 \text{ m/s}^2)}{0.5 \text{ kg/m}}} = 40 \text{ m/s} = 90 \text{ mi/hr}. \]

\[ \tau = \sqrt{\frac{m}{gb}} = \sqrt{\frac{80 \text{ kg}}{(10 \text{ m/s}^2)(0.5 \text{ kg/m})}} = 4 \text{ s}. \]

b) With given numbers, \( v = (40 \text{ m/s}) \tanh(t/4) \) and \( x = (160 \text{ m}) \ln \cosh(t/4). \)

At 90\% of \( v_T \), \( v = 36 \text{ m/s}. \) \[ t_{90} := \text{fsolve}(36 = 40 \tanh(t/4), t); \]
yields \( t_{90} = 5.9 \text{ s}. \)

Substituting this into the expression for \( x \), we get \( x = 130 \text{ m} = 430 \text{ ft}. \)

c) Let the ball radius be \( R \). From above analysis, \( b \) is proportional to \( A \) and hence to \( R^2 \). Mass \( m \) of ball is proportional to its volume, and hence to \( R^3 \). The terminal velocity \( v_T \) is proportional to \( \sqrt{m/b} \) and hence to \( \sqrt{R} \).