33-231 Physical Analysis Fall 2003

Problem Solutions: Set 2 (September 10, 2003)

6. a) Units of a are $m^{-1}s^{-2}$, i.e., $(\text{meters})^{-1} \times (\text{seconds})^{-2}$.

   b) $x = \dfrac{2}{at^2 + (\text{constant})}$.

   d) $x = \dfrac{2}{4t^2 + 1}$.

   g) $t = \pm 1/2$ s.

7. a) From $\Sigma = ma$, $\dfrac{dv}{dt} = -g - bv$. Separate variables and integrate:

$$\frac{dv}{v + \frac{mg}{b}} = -\frac{b}{m}dt, \quad \ln\left(v + \frac{mg}{b}\right) = -\frac{b}{m}t + \text{const.}$$

$$v + \frac{mg}{b} = e^{-(b/m)t + \text{const}} = (\text{const})\, e^{-(b/m)t}.$$

At time $t = 0$, $v_o + \dfrac{mg}{b} = \text{const.}$ Substitute and re-arrange to obtain

$$v = -\frac{mg}{b} + \left(v_o + \frac{mg}{b}\right)e^{-(b/m)t} = \frac{dy}{dt}.$$

Separate variables and integrate again to obtain

$$y = -\frac{mg}{b}t + \left(v_o + \frac{mg}{b}\right)\left(-\frac{m}{b}\right)e^{-(b/m)t} + \text{const.} \quad \text{At time } t = 0,$$

$$y_o = \left(v_o + \frac{mg}{b}\right)\left(-\frac{m}{b}\right) + \text{const.} \quad \text{Substitute into previous equation to obtain}$$

$$y = y_o - \frac{mg}{b}t + \frac{m}{b}\left(v_o + \frac{mg}{b}\right)\left(1 - e^{-(b/m)t}\right).$$

   b) See bboard post for Maple code.

   c) Substitute into previous results and simplify.

8. a) $e^{-(b/m)t} = 1 - \frac{b}{m}t + \frac{b^2t^2}{2m^2} + \cdots$.

b) $$v = -\frac{mg}{b} + \left(v_o + \frac{mg}{b}\right)\left(1 - \frac{b}{m}t + \frac{b^2t^2}{2m^2} + \cdots\right)$$

$$= v_o - gt - v_o\frac{b}{m}t + \frac{v_o b^2t^2}{2m^2} + \frac{mgb^2t^2}{2bm^2} + \cdots .$$

Because t is finite, the last three terms (and all successive terms) go to zero as $b \to 0$, leaving $v = v_o - gt$, as expected.

c) $$y = y_o - \frac{mg}{b}t + \frac{m}{b}\left(v_o + \frac{mg}{b}\right)\left[1 - \left(1 - \frac{b}{m}t + \frac{b^2t^2}{2m^2} + \cdots\right)\right].$$

Multiply out:

$$y = y_o - \frac{mg}{b}t + v_ot + \frac{mg}{b}t - \frac{v_ot^2b}{2m} - \frac{1}{2}gt^2.$$

The second and fourth terms cancel and the fifth (and all successive terms) go to zero as $b \to 0$, leaving

$$y = y_o + v_ot - \frac{1}{2}gt^2,$$ as expected.

9. a) $-bv^2 + mg = m\frac{dv}{dt}$. When $\frac{dv}{dt} = 0$, $v = v_T = \sqrt{\frac{mg}{b}}$.

The units of b are $\frac{\text{force}}{(\text{velocity})^2} = \frac{\text{kg m/s}^2}{\text{m}^2/\text{s}^2} = \frac{\text{kg}}{\text{m}}$. The units of v_T are then

$$\sqrt{\frac{\text{kg m/s}^2}{\text{kg/m}}} = \text{m/s},$$ as expected.

b) $v = \sqrt{\frac{mg}{b}}\tanh\sqrt{\frac{gb}{m}}t$. Let $\tau = \sqrt{\frac{m}{gb}}$; then $v = v_T\tanh\frac{t}{\tau}$.

The units of τ are $\sqrt{\frac{\text{kg}}{(\text{m/s}^2)(\text{kg/m})}} = \text{s}$,

(as required to make the quantity t/τ dimensionless).

(continued)

9. (continued

c) $$y = \frac{m}{b}\ln\left[\cosh\left(\sqrt{\frac{gb}{m}}t\right)\right] = v_T\tau\ln\left[\cosh\left(\frac{t}{\tau}\right)\right].$$

e) From Maple taylor command, the first term in the Taylor expansion of $v(t)$ reduces to gt. All other terms contain positive powers of b and therefore go to zero as $b \to 0$. Similarly, the first term in the Taylor series expansion of $y(t)$ is $gt^2/2$. All other terms contain positive powers of b.

10. a) $$b = \frac{CA\rho}{2} = \frac{(0.8)(1\text{ m}^2)(1.2\text{ kg/m}^3)}{2} = 0.5\ \text{kg/m}.$$

$$v_T = \sqrt{\frac{mg}{b}} = \sqrt{\frac{(80\text{ kg})(10\ \text{m/s}^2)}{0.5\text{kg / m}}} = 40\ \text{m/s} = 90\ \text{mi/hr}.$$

$$\tau = \sqrt{\frac{m}{gb}} = \sqrt{\frac{80\text{ kg}}{(10\ \text{m/s}^2)(0.5\ \text{kg/m})}} = 4\text{ s}.$$

b) With given numbers, $v = (40\text{ m/s})\tanh(t/4\text{s})$ and $x = (160\text{m})\ \ln\cosh(t/4\text{s})$.

At 90% of v_T, $v = 36$ m/s. `t90 := fsolve(36 = 40*tanh(t/4), t);` yields

$t_{90} = 5.9$ s. Substituting this into the expression for x , we get

$x = 130$ m = 430 ft.

c) Let the ball radius be R. From above analysis, b is proportional to A and hence to R^2. Mass m of ball is proportional to its volume, and hence to R^3. The terminal velocity v_T is proportional to $\sqrt{m/b}$ and hence to $\sqrt{R}$.