33-231 Physical Analysis Fall 2003

<u>Problem Solutions:</u> Set 13 (November 24, 2003)

51. a) $$\frac{\partial y}{\partial x} = -Ak\sin(kx+\omega t) - Ak\sin(kx-\omega t).$$

At $x = 0$, $\dfrac{\partial y}{\partial x} = -Ak\sin(\omega t) - Ak\sin(-\omega t) = 0$ because $\sin(-\alpha) = -\sin(\alpha)$

b) Use the cosine sum identities: $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$:

$$y = A\cos kx\cos\omega t - A\sin kx\sin\omega t + A\cos kx\cos\omega t + A\sin kx\sin\omega t$$

$$= 2A\cos kx\cos\omega t. \quad \text{Then re-define } A.$$

c) $y(x,t) = A\cos kx\cos\omega t.$ At $x = L$, $y(L, t)$ must be zero.

$$\cos kL = 0, \quad kL = \left(n + \frac{1}{2}\right)\pi, \quad n = 0, 1, 2, 3, \cdots.$$

$$\omega_n = ck_n = \frac{(n+\frac{1}{2})\pi c}{L}, \quad n = 0, 1, 2, 3, \cdots.$$

$$y(x,t) = A\cos\left[\frac{(n+\frac{1}{2})\pi x}{L}\right]\cos\left[\frac{(n+\frac{1}{2})\pi c t}{L}\right], \quad n = 0, 1, 2, 3, \cdots.$$

The frequencies are *odd* multiples of the fundamental :

$\omega_o = \dfrac{\pi c}{2L}$, $f_o = \dfrac{\omega_o}{2\pi} = \dfrac{c}{4L}$; this is half as large as when there are nodes at both ends, and n is the number of nodes (not counting the one at $x = 0$).

52. a) For (38), solve the simultaneous equations, Eqs. (36) and (37), by any method. (Maple is easiest.)

For (39): $k_1 = \omega/c_1$, $k_2 = \omega/c_2$. Substitute into Eq. (38):

$$B = \frac{\omega/c_1 - \omega/c_2}{\omega/c_1 + \omega/c_1}A = \frac{c_2 - c_1}{c_1 + c_2}A, \qquad C = \frac{2\omega/c_1}{\omega/c_1 + \omega/c_2}A = \frac{2c_2}{c_1 + c_2}A.$$

b) $c_1 = \sqrt{\dfrac{F}{\mu_1}}$, $c_2 = \sqrt{\dfrac{F}{\mu_2}}$. Substitute into Eq. (39):

$$B = \frac{\sqrt{(F/\mu_2)} - \sqrt{(F/\mu_1)}}{\sqrt{(F/\mu_1)} + \sqrt{(F/\mu_2)}}A = \frac{\sqrt{\mu_1} - \sqrt{\mu_2}}{\sqrt{\mu_1} + \sqrt{\mu_2}}A,$$

$$C = \frac{2\sqrt{(F/\mu_2)}}{\sqrt{(F/\mu_1)} + \sqrt{(F/\mu_2)}}A = \frac{2\sqrt{\mu_1}}{\sqrt{\mu_1} + \sqrt{\mu_2}}A.$$

53. a) Wave functions are the same as Eqs. (35). Boundary conditions at $x = 0$:

$$y_+ = y_-, \qquad A\cos(-\omega t) + B\cos(\omega t) = C\cos(-\omega t), \quad \text{and}$$

$$A + B = C.$$

$$F_1 \frac{\partial y_-}{\partial x} = F_2 \frac{\partial y_+}{\partial x}, \quad F_1\Big[-Ak_1 \sin(-\omega t) - Bk_1 \sin(\omega t)\Big] = -F_2 C k_2 \sin(-\omega t),$$

$$F_1 k_1 A - F_1 k_1 B = F_2 k_2 C.$$ Solve simultaneous equations for B and C,

then use $\dfrac{F_1}{c_1} = \dfrac{F_1}{\sqrt{F_1/\mu}} = \sqrt{\mu F_1}$:

$$B = \frac{F_1 k_1 - F_2 k_2}{F_1 k_1 + F_2 k_2} A = \frac{F_1/c_1 - F_2/c_2}{F_1/c_1 + F_2/c_2} A = \frac{\sqrt{F_1} - \sqrt{F_2}}{\sqrt{F_1} + \sqrt{F_2}} A,$$

$$C = \frac{2F_1 k_1}{F_1 k_1 + F_2 k_2} A = \frac{2\,F_1/c_1}{F_1/c_1 + F_2/c_2} A = \frac{2\sqrt{F_1}}{\sqrt{F_1} + \sqrt{F_2}} A.$$

Note that if both F and μ are different on the two sides, then

$$B = \frac{\sqrt{\mu_1 F_1} - \sqrt{\mu_2 F_2}}{\sqrt{\mu_1 F_1} + \sqrt{\mu_2 F_2}} A \qquad \text{and} \qquad C = \frac{2\sqrt{\mu_1 F_1}}{\sqrt{\mu_1 F_1} + \sqrt{\mu_2 F_2}} A.$$

The characteristic impedance of each side is $Z = \sqrt{\mu F}$. If $Z_1 = Z_2$, then there is no reflected wave.

54.

$$y(x,t) = A\cos(kx + \omega t), \qquad v_y = -A\omega \sin(kx + \omega t),$$

$$F_y = -F\frac{\partial y}{\partial x} = FAk \sin(kx + \omega t).$$

At $x = 0$, $\quad v_y = -A\omega \sin\omega t, \qquad F_y = FAk \sin\omega t,$

$$\frac{F_y}{v_y} = \frac{-FAk \sin\omega t}{A\omega \sin\omega t} = -\frac{Fk}{\omega} = -\frac{F}{c} = \frac{F}{\sqrt{F/\mu}} = \sqrt{\mu F}.$$

So $F_y = -bv_y$, where $b = \sqrt{\mu F}$. What we need is a shock absorber with a damping constant equal to b.