33-231 **Physical Analysis** **Fall 2003**

Problem Solutions: Set 11 (November 12, 2003)

43. a) $3m\ddot{x}_1 = -2kx_1 + kx_2$, $2m\ddot{x}_2 = kx_1 - kx_2$.

$$\mathbf{M} = \begin{pmatrix} 3m & 0 \\ 0 & 2m \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}, \quad \begin{pmatrix} 3m & 0 \\ 0 & 2m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

b) $$\mathbf{K} - \lambda\mathbf{M} = \begin{pmatrix} 2k - 3\lambda m & -k \\ -k & k - 2\lambda m \end{pmatrix},$$

$$|\mathbf{K} - \lambda\mathbf{M}| = \begin{vmatrix} 2k - 3\lambda m & -k \\ -k & k - 2\lambda m \end{vmatrix} = (2k - 3\lambda m)(k - 2\lambda m) - k^2 = 0.$$ Factor:

$$k^2 - 7k\lambda m + 6\lambda^2 m = (k - m\lambda)(k - 6\lambda m) = 0. \qquad \lambda_1 = \frac{k}{6m}, \quad \lambda_2 = \frac{k}{m}.$$

c) For λ_1: $$\mathbf{K} - \lambda_1\mathbf{M} = \begin{pmatrix} \frac{3}{2}k & -k \\ -k & \frac{2}{3}k \end{pmatrix}, \quad \tfrac{3}{2}ka_1 - ka_2 = 0, \quad \mathbf{a} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix},$$

$$(\mathbf{K} - \lambda_1\mathbf{M})\mathbf{a} = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{3}{2}k & -k \\ -k & \frac{2}{3}k \end{pmatrix}\begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix} = 0.$$

For λ_2: $$\mathbf{K} - \lambda_2\mathbf{M} = \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}, \quad -kb_1 - kb_2 = 0, \quad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$(\mathbf{K} - \lambda_2\mathbf{M})\mathbf{b} = 0 \quad \Rightarrow \quad \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.$$

d) $$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ \frac{3}{2} & -1 \end{pmatrix}, \quad \mathbf{x} = \mathbf{A}\mathbf{q}, \quad \begin{aligned} x_1 &= q_1 + q_2, \\ x_2 &= \frac{3}{2}q_1 - q_2, \end{aligned} \quad \mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} \\ \frac{3}{5} & -\frac{2}{5} \end{pmatrix}.$$

e) $$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}, \qquad \mathbf{M}\mathbf{A}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{A}\mathbf{q},$$

$$3m\ddot{q}_1 + 3m\ddot{q}_2 = -\frac{k}{2}q_1 - 3kq_2,$$

$$3m\ddot{q}_1 - 2m\ddot{q}_2 = -\frac{k}{2}q_1 + 2kq_2.$$

Add $2\times$ first equation to $3\times$ second equation:

$$15m\ddot{q}_1 = -\frac{5}{2}kq_1, \qquad m\ddot{q}_1 = -\frac{1}{6}kq_1, \qquad \omega_1^2 = \frac{k}{6m}.$$

Subtract second equation from first:

$$5m\ddot{q}_2 = -5kq_2, \qquad m\ddot{q}_2 = -kq_2, \qquad \omega_2^2 = \frac{k}{m}.$$

44. a) $3m\ddot{x}_1 = -kx_1 + kx_2$, $2m\ddot{x}_1 + 2m\ddot{x}_2 = -kx_2$.

$$\mathbf{M} = \begin{pmatrix} 3m & 0 \\ 2m & 2m \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} k & -k \\ 0 & k \end{pmatrix}, \quad \begin{pmatrix} 3m & 0 \\ 2m & 2m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k & -k \\ 0 & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

b) $$\mathbf{K} - \lambda\mathbf{M} = \begin{pmatrix} k-3\lambda m & -k \\ -2\lambda m & k-2\lambda m \end{pmatrix},$$

$$|\mathbf{K} - \lambda\mathbf{M}| = (k-3\lambda m)(k-2\lambda m) - 2k\lambda m = 0.$$

Expand and factor: $(k-\lambda m)(k-6\lambda m) = 0$, $\lambda_1 = \dfrac{k}{6m}$, $\lambda_2 = \dfrac{k}{m}$.

c) For λ_1: $\mathbf{K} - \lambda_1\mathbf{M} = \begin{pmatrix} \frac{1}{2}k & -k \\ -\frac{1}{3}k & \frac{2}{3}k \end{pmatrix}$, $\frac{1}{2}ka_1 - ka_2 = 0$, $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$(\mathbf{K} - \lambda_1\mathbf{M})\mathbf{a} = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{1}{2}k & -k \\ -\frac{1}{3}k & \frac{2}{3}k \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0.$$

For λ_2: $\mathbf{K} - \lambda_2\mathbf{M} = \begin{pmatrix} -2k & -k \\ -2k & -k \end{pmatrix}$, $-2kb_1 - kb_2 = 0$, $\mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$,

$$(\mathbf{K} - \lambda_2\mathbf{M})\mathbf{b} = 0 \quad \Rightarrow \quad \begin{pmatrix} -2k & -k \\ -2k & -k \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0.$$

d) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, $\mathbf{x} = \mathbf{A}\mathbf{q}$, $\begin{matrix} x_1 = 2q_1 + q_2 , \\ x_2 = q_1 - 2q_2 , \end{matrix}$ $\mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{5} \end{pmatrix}$.

e) $\mathbf{M}\,\ddot{\mathbf{x}} = -\mathbf{K}\,\mathbf{x}$, $\mathbf{M}\,\mathbf{A}\,\ddot{\mathbf{q}} = -\mathbf{K}\,\mathbf{A}\,\mathbf{q}$,

$$6m\ddot{q}_1 + 3m\ddot{q}_2 = -kq_1 - 3kq_2 ,$$
$$6m\ddot{q}_1 - 2m\ddot{q}_2 = -kq_1 + 2kq_2 .$$

Add $2\times$ first equation to $3\times$ second equation:

$$30m\ddot{q}_1 = -5kq_1 , \qquad m\ddot{q}_1 = -\frac{k}{6}q_1, \qquad \omega_1^{\,2} = \frac{k}{6m}.$$

Subtract second equation from first:

$$5m\ddot{q}_2 = -5kq_2 , \qquad m\ddot{q}_2 = -kq_2 , \qquad \omega_2^{\,2} = \frac{k}{m}.$$

45. a) $m\ddot{x}_1 = -kx_1 + kx_2,\qquad m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3,\qquad m\ddot{x}_3 = kx_2 - kx_3.$

$$\mathbf{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} = k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

$$\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} = -\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

b) $$\mathbf{K} - \lambda\mathbf{M} = \begin{pmatrix} k-\lambda m & -k & 0 \\ -k & 2k-\lambda m & -k \\ 0 & -k & k-\lambda m \end{pmatrix}.$$

$$|\mathbf{K} - \lambda\mathbf{M}| = \begin{vmatrix} k-\lambda m & -k & 0 \\ -k & 2k-\lambda m & -k \\ 0 & -k & k-\lambda m \end{vmatrix} = (k-\lambda m)^2(2k-\lambda m) - 2k^2(k-\lambda m) = 0.$$

Factor, multiply out, and factor again: $|\mathbf{K} - \lambda\mathbf{M}| = (-\lambda m)(k-\lambda m)(3k-\lambda m) = 0.$

$$\lambda_1 = 0, \qquad \lambda_2 = \frac{k}{m}, \qquad \lambda_3 = \frac{3k}{m}.$$

c) For λ_1: $$\mathbf{K} - \lambda_1\mathbf{M} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}, \qquad \begin{aligned} ka_1 - ka_2 &= 0, \\ -ka_1 + 2ka_2 - ka_3 &= 0, \\ -ka_2 + ka_3 &= 0, \end{aligned} \qquad \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For λ_2, $$\mathbf{K} - \lambda_2\mathbf{M} = \begin{pmatrix} 0 & -k & 0 \\ -k & k & -k \\ 0 & -k & 0 \end{pmatrix}, \qquad \begin{aligned} -b_2 &= 0, \\ -b_1 + b_2 - b_3 &= 0, \\ -b_2 &= 0. \end{aligned} \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For λ_3, $$\mathbf{K} - \lambda_3\mathbf{M} = \begin{pmatrix} -2k & -k & 0 \\ -k & -k & -k \\ 0 & -k & -2k \end{pmatrix}, \qquad \begin{aligned} -2c_1 - c_2 &= 0, \\ -c_1 - c_2 - c_3 &= 0, \\ -c_2 - 2c_3 &= 0. \end{aligned} \qquad \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

(continued)

45. (continued)

d)

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}, \qquad \mathbf{x} = \mathbf{A}\mathbf{q}, \qquad \begin{aligned} x_1 &= q_1 + q_2 + q_3, \\ x_2 &= q_1 - 2q_3, \\ x_3 &= q_1 - q_2 + q_3. \end{aligned} \qquad \mathbf{A}^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}.$$

e)

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}, \qquad \mathbf{M}\mathbf{A}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{A}\mathbf{q},$$

$$m\ddot{q}_1 + m\ddot{q}_2 + m\ddot{q}_3 = -kq_2 - 3kq_3,$$

$$m\ddot{q}_1 - 2m\ddot{q}_3 = 6kq_3,$$

$$m\ddot{q}_1 - m\ddot{q}_2 + m\ddot{q}_3 = kq_2 - 3kq_3.$$

Adding the three equations gives $3m\ddot{q}_1 = 0, \qquad \lambda_1 = \omega_1^{\,2} = 0.$

The first minus the third gives: $2m\ddot{q}_2 = -2kq_2, \qquad \lambda_2 = \omega_2^{\,2} = \dfrac{k}{m}.$

First plus third minus $2 \times$ second: $6m\ddot{q}_3 = -18kq_3, \qquad \lambda_3 = \omega_3^{\,2} = \dfrac{3k}{m}.$