Problem Solutions: Set 11 (November 12, 2003)

43. a) \(3m\ddot{x}_1 = -2kx_1 + kx_2, \quad 2m\ddot{x}_2 = kx_1 - kx_2.\)

\[
M = \begin{pmatrix} 3m & 0 \\ 0 & 2m \end{pmatrix}, \quad K = \begin{pmatrix} 2k & -k \\ -k & k-2\lambda m \end{pmatrix}, \quad \begin{pmatrix} 3m \\ 0 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = -\begin{pmatrix} 2k & -k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
\]

b) \(K - \lambda M = \begin{pmatrix} 2k - 3\lambda m & -k \\ -k & k - 2\lambda m \end{pmatrix},\)

\[
|K - \lambda M| = \begin{vmatrix} 2k - 3\lambda m & -k \\ -k & k - 2\lambda m \end{vmatrix} = (2k - 3\lambda m)(k - 2\lambda m) - k^2 = 0. \quad \text{Factor:}
\]

\(k^2 - 7k\lambda m + 6\lambda^2 m = (k - m\lambda)(k - 6\lambda m) = 0. \quad \lambda_1 = \frac{k}{6m}, \quad \lambda_2 = \frac{k}{m}.
\]

c) For \(\lambda_1:\)

\[
(K - \lambda_1 M)\mathbf{a} = 0 \quad \Rightarrow \quad \begin{pmatrix} \frac{1}{2}k & -k \\ -k & \frac{1}{2}k \end{pmatrix} \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} = 0.
\]

For \(\lambda_2:\)

\[
(K - \lambda_2 M)\mathbf{b} = 0 \quad \Rightarrow \quad \begin{pmatrix} -k & -k \\ -k & -k \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0.
\]

d) \(A = \begin{pmatrix} \frac{1}{3} & 1 \\ \frac{1}{2} & -1 \end{pmatrix},\quad \mathbf{x} = A\mathbf{q}, \quad x_1 = q_1 + q_2, \quad x_2 = \frac{3}{2}q_1 - q_2, \quad A^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{pmatrix}.
\]

e) \(M\ddot{\mathbf{x}} = -K\mathbf{x}, \quad M\ddot{\mathbf{q}} = -K\mathbf{q}.
\]

\(3m\ddot{q}_1 + 3m\ddot{q}_2 = -\frac{k}{2}q_1 - 3kq_2,\)

\(3m\ddot{q}_1 - 2m\ddot{q}_2 = -\frac{k}{2}q_1 + 2kq_2.
\]

Add \(2 \times \) first equation to \(3 \times \) second equation:

\[
15\ddot{q}_1 = -\frac{5}{2}kq_1, \quad m\ddot{q}_1 = -\frac{1}{6}kq_1, \quad \omega_1^2 = \frac{k}{6m}.
\]

Subtract second equation from first:

\[
5m\ddot{q}_2 = -5kq_2, \quad m\ddot{q}_2 = -kq_2, \quad \omega_2^2 = \frac{k}{m}.
\]
44. a) $3m\ddot{x}_1 = -kx_1 + kx_2$, \hspace{1em} $2m\ddot{x}_1 + 2m\ddot{x}_2 = -kx_2$.

$$
\mathbf{M} = \begin{pmatrix} 3m & 0 \\ 2m & 2m \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} k & -k \\ 0 & k \end{pmatrix}, \quad \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = -\begin{pmatrix} k & -k \\ 0 & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.
$$

b) $\mathbf{K} - \lambda \mathbf{M} = \begin{pmatrix} k - 3\lambda m & -k \\ -2\lambda m & k - 2\lambda m \end{pmatrix}$,

$$|\mathbf{K} - \lambda \mathbf{M}| = (k - 3\lambda m)(k - 2\lambda m) - 2k\lambda m = 0.$$ Expand and factor: $(k - \lambda m)(k - 6\lambda m) = 0$, \hspace{1em} $\lambda_1 = \frac{k}{6m}$, \hspace{1em} $\lambda_2 = \frac{k}{m}$.

c) For $\lambda_1$: $\mathbf{K} - \lambda_1 \mathbf{M} = \begin{pmatrix} \frac{2}{3}k & -k \\ -\frac{1}{3}k & 2\frac{2}{3}k \end{pmatrix}$, \hspace{1em} $\frac{1}{2}ka_1 - ka_2 = 0$, \hspace{1em} $a = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$$(\mathbf{K} - \lambda_1 \mathbf{M})a = 0 \implies \begin{pmatrix} \frac{2}{3}k & -k \\ -\frac{1}{3}k & 2\frac{2}{3}k \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0.$$

For $\lambda_2$: $\mathbf{K} - \lambda_2 \mathbf{M} = \begin{pmatrix} -2k & -k \\ 2k & -k \end{pmatrix}$, \hspace{1em} $-2k a_1 - kb_2 = 0$, \hspace{1em} $b = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$.

$$(\mathbf{K} - \lambda_2 \mathbf{M})b = 0 \implies \begin{pmatrix} -2k & -k \\ 2k & -k \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0.$$

d) $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$, \hspace{1em} $\mathbf{x} = \mathbf{A} \mathbf{q}$, \hspace{1em} $x_1 = 2q_1 + q_2$, \hspace{1em} $x_2 = q_1 - 2q_2$, \hspace{1em} $\mathbf{A}^{-1} = \begin{pmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{pmatrix}$.

e) $\mathbf{M} \ddot{\mathbf{x}} = -\mathbf{K} \mathbf{x}$, \hspace{1em} $\mathbf{M} \mathbf{A} \ddot{\mathbf{q}} = -\mathbf{K} \mathbf{A} \mathbf{q}$,

$6m\ddot{q}_1 + 3m\ddot{q}_2 = -kq_1 - 3kq_2$.

$6m\ddot{q}_1 - 2m\ddot{q}_2 = -kq_1 + 2kq_2$.

Add $2 \times$ first equation to $3 \times$ second equation:

$30m\ddot{q}_1 = -5kq_1$, \hspace{1em} $\ddot{q}_1 = -\frac{k}{6}q_1$, \hspace{1em} $\omega_1^2 = \frac{k}{6m}$.

Subtract second equation from first:

$5m\ddot{q}_2 = -5kq_2$, \hspace{1em} $\ddot{q}_2 = -kq_2$, \hspace{1em} $\omega_2^2 = \frac{k}{m}$. 
45. a) \(m \ddot{x}_1 = -k x_1 + k x_2, \quad m \ddot{x}_2 = k x_1 - 2k x_2 + k x_3, \quad m \ddot{x}_3 = k x_2 - k x_3.\)

\[
\mathbf{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} = m \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} = k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}
\]

\[
\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = -\begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.
\]

\[
\mathbf{K} - \lambda \mathbf{M} = \begin{pmatrix} k - \lambda m & -k & 0 \\ -k & 2k - \lambda m & -k \\ 0 & -k & k - \lambda m \end{pmatrix}
\]

\[
|\mathbf{K} - \lambda \mathbf{M}| = \begin{vmatrix} k - \lambda m & -k & 0 \\ -k & 2k - \lambda m & -k \\ 0 & -k & k - \lambda m \end{vmatrix} = (k - \lambda m)^2 (2k - \lambda m) - 2k^2 (k - \lambda m) = 0.
\]

Factor, multiply out, and factor again: \(|\mathbf{K} - \lambda \mathbf{M}| = (-\lambda m)(k - \lambda m)(3k - \lambda m) = 0.\)

\[
\lambda_1 = 0, \quad \lambda_2 = \frac{k}{m}, \quad \lambda_3 = \frac{3k}{m}.
\]

b) \(\mathbf{K} = \lambda \mathbf{M} = \begin{pmatrix} k - \lambda m & -k & 0 \\ -k & 2k - \lambda m & -k \\ 0 & -k & k - \lambda m \end{pmatrix}.
\]

For \(\lambda_1:\) \(\mathbf{K} - \lambda_1 \mathbf{M} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}, \quad -ka_1 - ka_2 = 0, \quad ka_1 + 2ka_2 - ka_3 = 0, \quad a = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]

For \(\lambda_2,\) \(\mathbf{K} - \lambda_2 \mathbf{M} = \begin{pmatrix} 0 & -k & 0 \\ -k & k & -k \\ 0 & -k & 0 \end{pmatrix}, \quad -b_1 = 0, \quad -b_1 + b_2 - b_3 = 0, \quad b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.
\]

For \(\lambda_3,\) \(\mathbf{K} - \lambda_3 \mathbf{M} = \begin{pmatrix} 0 & -k & 0 \\ -k & k & -k \\ 0 & -k & -2k \end{pmatrix}, \quad -c_1 - c_2 = 0, \quad -2c_1 - c_2 - c_3 = 0, \quad c = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.
\]
45. (continued)

d) \[ A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}, \quad x = Aq, \quad x_1 = q_1 + q_2 + q_3, \quad x_2 = q_1 - 2q_3, \quad x_3 = q_1 - q_2 + q_3. \]
\[ A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & -\frac{1}{2} \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} \end{pmatrix}. \]

e) \[ \mathbf{M} \ddot{x} = -\mathbf{K} x, \quad \mathbf{M} \mathbf{A} \ddot{q} = -\mathbf{K} \mathbf{A} \dot{q}, \]

\[
\begin{align*}
m\dddot{q}_1 + m\dddot{q}_2 + m\dddot{q}_3 &= -kq_2 - 3kq_3, \\
m\dddot{q}_1 - 2m\dddot{q}_3 &= 6kq_3, \\
m\dddot{q}_1 - m\dddot{q}_2 + m\dddot{q}_3 &= kq_2 - 3kq_3.
\end{align*}
\]

Adding the three equations gives: \[ 3m\dddot{q}_1 = 0, \quad \lambda_1 = \omega_1^2 = 0. \]

The first minus the third gives: \[ 2m\dddot{q}_2 = -2kq_2, \quad \lambda_2 = \omega_2^2 = \frac{k}{m}. \]

First plus third minus \( 2 \times \) second: \[ 6m\dddot{q}_3 = -18kq_3, \quad \lambda_3 = \omega_3^2 = \frac{3k}{m}. \]