

### References

- Chabay and Sherwood *Matter and Interactions*, Sections 3.4, 5.11, 5.13  
Young and Freedman *University Physics* (10th or 11th edition), Chapter 13  
Stewart *Calculus - Early Transcendentals* (4th edition), Sections 9.1 through 9.4, 9.6, 17.1 through 17.3  
Edwards and Penney *Differential Equations and Boundary Value Problems* (2nd edition), Sections 3.4, 3.6

### Periodic Motion

Any motion that repeats itself over and over with a definite cycle is said to be *periodic*. The rate of repetition is described by the *period* ( $T$ ), the *frequency* ( $f$ ), or the *angular frequency* ( $\omega$ ), defined as follows:

$$\begin{aligned} T &= \text{period} = \text{amount of time for one cycle;} \\ f &= \text{frequency} = \text{number of cycles per unit time;} \quad f = 1/T \\ \omega &= \text{angular frequency} = 2\pi f. \end{aligned}$$

The simplest example of periodic motion in a mechanical system is the *harmonic oscillator*; its motion is called *simple harmonic motion* (SHM). The defining characteristics of SHM are a stable equilibrium position and a restoring force that is *directly proportional* to the displacement from that position.

In many systems the restoring force is *approximately* proportional to the displacement from equilibrium; the resulting motion is then approximately simple harmonic. Thus SHM is useful as a *model* that describes the behavior of such systems approximately but not exactly. (But note the caution on page 4-5, concerning exceptional cases where  $F$  is *not* proportional to  $x$ , even for very small displacements.)

### Undamped Harmonic Motion

In its simplest form the harmonic oscillator consists of a mass  $m$  that moves along a straight line with coordinate  $x$ , under the action of a force  $F$  that acts along this line and has magnitude proportional to  $x$ , with proportionality constant  $k$ . (For example, a spring that obeys Hooke's law) At  $x = 0$ ,  $F = 0$ , so  $x = 0$  is a position of stable equilibrium. The force can be represented analytically as  $F = -kx$ . The negative sign shows that the direction of  $F$  is always toward the point  $x = 0$ , opposite to the displacement. The constant  $k$  is called the *force constant* or *spring constant* for the system.

When  $F = -kx$ , Newton's second law ( $\Sigma F = ma$ ) gives

$$-kx = m \frac{d^2x}{dt^2} \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}x. \quad (1)$$

In these equations (called *differential equations* because they contain derivatives),  $x$  must be some function of  $t$  such that when this function and its second derivative are substituted into either equation, the left and right sides are really equal. Two possibilities are

$$x = \sin \sqrt{\frac{k}{m}} t \quad \text{and} \quad x = \cos \sqrt{\frac{k}{m}} t. \quad (2)$$

These functions are said to be *solutions* of the differential equation. You should verify this by substitution.

Any differential equation that contains  $x$  and its derivatives only to the first power (called a *linear* differential equation) has the property that every linear combination of solutions is also a solution. Thus a more general solution is

$$x = A \cos \sqrt{\frac{k}{m}} t + B \sin \sqrt{\frac{k}{m}} t, \quad (3)$$

where  $A$  and  $B$  can be any constants. It can be shown that this is the *most general* solution of the differential equation, and therefore represents all the possible motions of the system. The general solution may also be written in the alternative form

$$x = C \cos \left( \sqrt{\frac{k}{m}} t + \phi \right). \quad (4)$$

We invite you to prove the equivalence of Eqs. (3) and (4). Use the identity for the cosine of the sum of two angles, to show that the forms are identical if

$$A = C \cos \phi, \quad B = -C \sin \phi, \quad \text{or} \quad C = \sqrt{A^2 + B^2}, \quad \phi = \arctan \left( \frac{-B}{A} \right). \quad (5)$$

Each of these functions goes through one cycle when the argument  $\sqrt{\frac{k}{m}} t$  goes from zero to  $2\pi$ , i.e., in a time  $T$  (the *period* of the motion) such that

$$\sqrt{\frac{k}{m}} T = 2\pi \quad \text{or} \quad T = 2\pi \sqrt{\frac{m}{k}}. \quad (6)$$

This result shows that greater mass  $m$  means more time for one period, while greater force constant  $k$  means less time for one period as we might have predicted.

From the definitions of  $f$  and  $\omega$ , we also find

$$f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega = \frac{2\pi}{T} = \sqrt{\frac{k}{m}}. \quad (7)$$

Anticipating later developments, we will use the notation  $\omega_o = \sqrt{k/m}$  (rather than  $\omega$ ) for angular frequency. In terms of  $\omega_o$ , the possible motions can be written as

$$x = A \cos \omega_o t + B \sin \omega_o t \quad \text{or} \quad x = C \cos (\omega_o t + \phi). \quad (8)$$

The constants  $A$  and  $B$ , or  $C$  and  $\phi$ , are determined by the initial conditions of the system, i.e., the position  $x_o$  and velocity  $v_o$  at the initial time  $t = 0$ . Specifically,

$$A = x_o, \quad B = \frac{v_o}{\omega_o} \quad \text{or} \quad C = \sqrt{x_o^2 + \frac{v_o^2}{\omega_o^2}}, \quad \phi = \arctan\left(\frac{-v_o}{\omega_o x_o}\right). \quad (9)$$

You should derive these relations, to verify the equivalence of the two forms of Eq. (8)

### Damped Harmonic Motion

*Damping* is the presence of an additional force, frictional and dissipative in nature, that causes the oscillations of the system to die out. The motion is then not strictly periodic; each cycle has somewhat smaller amplitude than the preceding one. When the frequency remains constant as the amplitude decreases, the motion is said to be *quasi-periodic*.

The simplest case to treat analytically is a viscous friction force that is proportional to the speed  $v = \dot{x}$  of the mass:  $F = -bv$ , where  $b$  is a constant that describes the strength of the damping force. (The negative sign shows that  $F$  always opposes the motion.)

The Newton's second law equation ( $\Sigma F = ma$ ) then becomes

$$-kx - b \frac{dx}{dt} = m \frac{d^2x}{dt^2}, \quad \text{or} \quad \frac{d^2x}{dt^2} + \frac{b}{m} \frac{dx}{dt} + \frac{k}{m} x = 0. \quad (10)$$

Using the abbreviation  $\gamma = \frac{b}{2m}$  along with  $\omega_o^2 = \frac{k}{m}$  and the abbreviations

$$\frac{dx}{dt} = \dot{x} \quad \text{and} \quad \frac{d^2x}{dt^2} = \ddot{x} \quad (\text{a notation introduced by Newton}),$$

we can write this differential equation compactly as

$$\ddot{x} + 2\gamma\dot{x} + \omega_o^2 x = 0. \quad (11)$$

Solution of Equation (11) is discussed in detail in Stewart and in Edwards and Penney.

We substitute a trial solution in the form  $x = e^{pt}$ . This is a solution of Eq. (11) if  $p$  satisfies the characteristic equation  $p^2 + 2\gamma p + \omega_o^2 = 0$ . This equation has two unequal real roots if  $\gamma > \omega_o$ , two complex roots if  $\gamma < \omega_o$ , and two equal real roots if  $\gamma = \omega_o$ . We'll discuss these three cases separately.

### Overdamping

In the first case ( $\gamma > \omega_0$ ), we introduce the abbreviation  $\gamma_d = \sqrt{\gamma^2 - \omega_0^2}$ ; then the most general solution is

$$x = Ae^{-(\gamma+\gamma_d)t} + Be^{-(\gamma-\gamma_d)t} = e^{-\gamma t} (Ae^{-\gamma_d t} + Be^{\gamma_d t}), \quad (12)$$

where  $A$  and  $B$  are constants. This solution can be verified by direct substitution using Maple. Because  $\gamma_d$  is always less than  $\gamma$ , both terms in Eq. (12) are always *decaying* exponentials, with no oscillation. When  $\gamma$  is greater than  $\omega_0$ , the system is said to be *overdamped*. If the initial conditions (at time  $t = 0$ ) are  $x_0$  and  $v_0$ , then

$$A = -\frac{(\gamma - \gamma_d)x_0 + v_0}{2\gamma_d}, \quad B = \frac{(\gamma + \gamma_d)x_0 + v_0}{2\gamma_d}. \quad (13)$$

### Underdamping

In the case  $\gamma < \omega_0$  we define  $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$ ; then the most general solution is

$$x = e^{-\gamma t} (A \cos \omega_d t + B \sin \omega_d t), \quad (14)$$

where  $A$  and  $B$  are constants. This solution can be verified by direct substitution using Maple. It represents a decaying oscillation with an angular frequency  $\omega_d$  that is *less than*  $\omega_0$ , with exponentially decaying amplitude. Such a system is said to be *underdamped*, and the motion is quasi-periodic. In terms of the initial conditions  $x_0$  and  $v_0$  at time  $t = 0$ ,  $A$  and  $B$  are given by

$$A = x_0, \quad B = \frac{v_0 + \gamma x_0}{\sqrt{\omega_0^2 - \gamma^2}} = \frac{v_0 + \gamma x_0}{\omega_d}. \quad (15)$$

Note that if  $\gamma = 0$ , these expressions reduce to those for the undamped case (Eqs. (9)), and also that in this case  $\omega_d = \omega_0$ .

### Critical Damping

A special case occurs when the damping is just large enough so that  $\gamma = \omega_0$ . Then the characteristic equation has a double root, and the general solution is

$$x = (A + Bt)e^{-\gamma t}. \quad (16)$$

The constants  $A$  and  $B$  are again determined by the initial conditions  $x_0$  and  $v_0$ :

$$A = x_0, \quad B = \gamma x_0 + v_0. \quad (17)$$

This condition is called *critical damping*. There is no oscillation, but the approach to equilibrium is faster than with overdamping.

The reader is urged to verify the solutions given by Eqs. (12), (14), and (16), and to derive the initial-condition relations given by Eqs. (13), (15), and (17).

**Energy Relations for Undamped Oscillator**

The kinetic energy  $K$  and potential energy  $V$  for the harmonic oscillator are given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{x}^2, \quad V = \frac{1}{2}kx^2. \quad (18)$$

For the *undamped* oscillator, the force is conservative, and the total energy  $E = K + V$  (kinetic plus potential) is constant. To show this, we calculate the time derivative of  $E$ :

$$\frac{dE}{dt} = \frac{d}{dt}(K + V) = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx). \quad (19)$$

Because of Eq. (1),  $m\ddot{x} + kx = 0$ , so  $dE/dt = 0$  and the total energy  $E$  is constant.

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 = \text{constant}. \quad (20)$$

If we plot a graph with  $x$  on the horizontal axis and  $\dot{x} = v$  on the vertical axis, Eq. (20) is the equation of an *ellipse*. The two-dimensional space of this graph is called a *phase space*, and the plot is called a *phase plot*. Each point in the  $x$ - $v$  plane represents an instantaneous state of the particle (position and velocity). As the motion progresses, the representative point traces out a curve called the *phase trajectory* in this plane. For undamped SHM, the phase trajectory is an ellipse in the  $x$ - $v$  plane.

If the motion of the system is described by Eq. (8), then it can be shown that the total energy is given by

$$E = \frac{1}{2}k(A^2 + B^2) = \frac{1}{2}kC^2. \quad (21)$$

**Energy Relations and  $Q$  for Damped Oscillator**

When a damping force  $F = -bv$  is present, the total energy is no longer constant. The rate of change of total energy is still given by Eq. (19), but it is not zero. Rewriting Eq. (10) as  $m\ddot{x} + kx = -b\dot{x}$  and substituting into Eq. (19), we find

$$\frac{dE}{dt} = -b\dot{x}^2 = -(b\dot{x})\dot{x} = -b\dot{x}^2 \quad (22)$$

This has a simple physical interpretation; when a force  $F$  acts on a body moving with velocity  $v$  in the direction of the force, the power (rate of doing work) is  $Fv$ . Here the damping force is  $-b\dot{x}$ , so Eq. (22) represents the (negative) rate at which the damping force does work on the system, equal to the rate of change of its total energy. Note that  $dE/dt$  is never positive; the energy continuously decreases.

It is of interest to compare the energy loss *in one cycle*,  $\Delta E$ , for the underdamped oscillator, to the energy  $E$  at the beginning of that cycle. We define a quantity  $Q$ :

$$Q = 2\pi \left| \frac{E}{\Delta E} \right|. \quad (23)$$

Larger values of  $Q$  correspond to weaker damping, smaller  $\Delta E$ , and more slowly decaying oscillations, and conversely. (In remote antiquity, when this analysis was applied to  $L$ - $C$  resonant circuits in radio equipment,  $Q$  was an abbreviation for *quality factor*. It is a measure of the sharpness of a resonance peak in the circuit. More about resonance later.)

### Light Damping Approximation

We can derive a relation of  $Q$  to the system parameters  $\omega_o$  and  $\gamma$ . We'll limit our discussion to systems with *light damping*. In such systems, the maximum displacements from equilibrium change relatively little from one cycle to the next and the oscillations die out gradually over many cycles. Then the exponential factor in Eq. (14) changes by only a small fraction of its value during one cycle. The *period* of the damped oscillation is  $2\pi/\omega_d$ , so the condition for light damping is

$$\frac{2\pi\gamma}{\omega_d} \ll 1, \quad \text{or} \quad \gamma \ll \omega_d \quad \text{and} \quad \gamma \ll \omega_o, \quad \text{so} \quad \omega_d \cong \omega_o. \quad (24)$$

In the light-damping approximation,  $\omega_d$  is very nearly equal to the undamped angular frequency  $\omega_o$ . For such a system we expect that  $\Delta E$  will be much smaller than  $E$ , so the quantity  $Q$  defined by Eq. (23) will be much larger than unity. The following discussion will confirm this expectation.

Suppose the motion is given by  $x = Ae^{-\gamma t} \cos \omega_d t$ . Then the maxima and minima of  $x$  occur approximately when  $\omega_d t = n\pi$ , where  $n$  is an integer. At these points, the body is at rest and the energy is entirely potential. In particular, at time  $t = 0$ , the total energy is  $E_o = \frac{1}{2}kA^2$ , and at the end of one period, when  $\omega_d t = 2\pi$  and  $t = 2\pi/\omega_d$ ,

$$E_1 = \frac{1}{2}k\left(Ae^{-(2\pi\gamma/\omega_d)}\right)^2 = \frac{1}{2}kA^2 e^{-4\pi\gamma/\omega_d}. \quad (25)$$

The fractional change of energy during one cycle is then (apart from sign)

$$\left|\frac{\Delta E}{E}\right| = \frac{E_o - E_1}{E_o} = 1 - \frac{E_1}{E_o} = 1 - e^{-4\pi\gamma/\omega_d}. \quad (26)$$

In the light damping approximation,  $\gamma$  is much smaller than  $\omega_d$ , so we can expand the exponential function in Eq. (26) in a Taylor series, keeping only the first two terms:

$$e^{-4\pi\gamma/\omega_d} \cong 1 - \frac{4\pi\gamma}{\omega_d}, \quad (27)$$

and Eq. (26) becomes approximately

$$\left|\frac{\Delta E}{E}\right| \cong \frac{4\pi\gamma}{\omega_d} \cong \frac{4\pi\gamma}{\omega_o}. \quad (28)$$

Finally, we substitute this result into the definition of  $Q$ , Eq. (23):

$$Q = 2\pi \left|\frac{E}{\Delta E}\right| = 2\pi \left(\frac{\omega_o}{4\pi\gamma}\right) = \frac{\omega_o}{2\gamma}. \quad (29)$$

We also note that  $|\Delta E/2\pi|$  represents the energy loss *per radian* (i.e., per  $1/2\pi$  cycles). Thus an alternative definition of  $Q$  is

$$Q = \left| \frac{E}{\Delta E_{\text{per radian}}} \right|.$$

The above analysis shows that in the light-damping approximation, if the total energy  $E$  is measured at corresponding points on successive cycles, it decays exponentially according to

$$E = E_0 e^{-2\gamma t}, \quad (30)$$

where  $E_0$  is the energy at the initial time  $t = 0$ . Equation (29) also shows that a lightly damped system, for which  $\gamma \ll \omega_0$ , has a value of  $Q$  much larger than unity.

The rate of energy loss varies during each cycle; it is greatest when  $v$  is greatest (i.e., at  $x = 0$ ), and zero at points of maximum displacement (i.e., when  $v = 0$ ). The phase trajectory for underdamped oscillations is not an ellipse but a spiral that curves in toward the origin of the  $x$ - $v$  plane.

### **Forced Oscillations -- Undamped System**

A situation of great practical interest occurs when an additional time-dependent force  $F(t)$  (which we will call a *driving force*) is applied to a harmonic oscillator. An important special case is a driving force that varies *sinusoidally* with time, which we may represent as

$$F(t) = F_0 \cos \omega t. \quad (31)$$

In this expression,  $F_0$  is a constant that characterizes the strength of the driving force, and  $\omega$  is its angular frequency. Note that in general  $\omega$  is *not* equal to either  $\omega_0$  or  $\omega_d$  in the preceding discussion.

The additional force enables the system to undergo a motion called *forced* or *driven oscillation* that would not be possible without it. In particular, the system can move with an undamped sinusoidal motion having the same angular frequency as the driving force.

If there is a driving force given by Eq. (31), and no damping force, the equation of motion (from  $\Sigma F = ma$ ) is

$$-kx + F_0 \cos \omega t = m \frac{d^2 x}{dt^2}, \quad \text{or} \quad \ddot{x} + \omega_0^2 x = \frac{F_0}{m} \cos \omega t. \quad (32)$$

It is natural to look for a solution to this equation that has the same frequency as the driving force. Therefore we try a solution in the form  $x = A' \cos \omega t$ , where  $A'$  is a constant to be determined. Substituting this into Eq. (32), we find that it is a solution only if

$$A' = \frac{F_0/m}{\omega_0^2 - \omega^2}; \quad \text{then} \quad x = \frac{F_0/m}{\omega_0^2 - \omega^2} \cos \omega t. \quad (33)$$

The amplitude  $A'$  of the forced oscillation is proportional to the strength  $F_o$  of the driving force, as we might have expected. It is strongly dependent on the angular frequency  $\omega$  of the driving force; when this frequency equals the angular frequency  $\omega_o$  of *free* oscillations of the system,  $A'$  is infinite! This critical dependence of the amplitude on the angular frequency of the driving force is called *resonance*. Note that  $A'$  is *not* arbitrary, and that it is *not* determined by initial conditions, unlike the constants  $A$ ,  $B$ , and  $C$  in the previous analysis.

When  $\omega < \omega_o$ ,  $A'$  is positive and the forced oscillation is *in phase* with the driving force; the two have the same sign at each instant. When  $\omega > \omega_o$ ,  $A'$  is negative, and the two are a half-cycle out of phase; they have opposite signs at each instant.

In the language of differential equations, Eq. (33) is a *particular solution* of Eq. (32), but it is not the *complete* solution. That is obtained by adding to Eq. (33) the general solution of the corresponding homogeneous equation, which is given by Eq. (8). Thus the most general solution of Eq. (32) is

$$x = A \cos \omega_o t + B \sin \omega_o t + \frac{F_o/m}{\omega_o^2 - \omega^2} \cos \omega t. \quad (34)$$

When a forcing function is present, Eqs.(9) are no longer valid. To obtain the relation of the constants  $A$  and  $B$  to the initial conditions  $x_o$  and  $v_o$ , it is essential to use the *complete* solution, Eq. (34).

### **Forced Oscillations -- Damped System**

When the sinusoidal driving force of Eq. (31) acts on a *damped* harmonic oscillator the equation of motion (from  $\Sigma F = ma$ ) is

$$-kx - b \frac{dx}{dt} + F_o \cos \omega t = m \frac{d^2 x}{dt^2}, \quad \text{or} \quad \ddot{x} + 2\gamma \dot{x} + \omega_o^2 x = \frac{F_o}{m} \cos \omega t. \quad (35)$$

Again we look for a sinusoidal solution representing a forced oscillation with angular frequency  $\omega$ . Because Eq. (35) now contains both first and second derivatives, the solution will contain both  $\cos \omega t$  and  $\sin \omega t$ , or (alternatively) a cosine function with a phase angle. Hence we try a particular solution in the form

$$x = A' \cos(\omega t + \varphi), \quad (36)$$

where  $A'$  and  $\varphi$  are constants to be determined.

When Eq. (36) is substituted into Eq. (35), it is found to satisfy that equation only when

$$A' = \frac{F_o/m}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2\gamma\omega)^2}}, \quad \tan \varphi = \frac{2\gamma\omega}{\omega^2 - \omega_o^2}. \quad (37)$$

The derivation of these results is straightforward but somewhat complicated. We postpone the derivation of Eqs. (37) until page 12.



The denominator in the  $A'$  expression is often abbreviated as  $D(\omega)$ ,

$$D(\omega) = \sqrt{(\omega^2 - \omega_o^2)^2 + (2\gamma\omega)^2}, \quad \text{so} \quad A'(\omega) = \frac{F_o/m}{D(\omega)}. \quad (38)$$

The dependence of  $A'$  and  $D$  on  $\omega$  is written explicitly as a reminder that the amplitude of the forced oscillation again depends critically on the angular frequency  $\omega$  of the driving force

### Resonance

A graph of  $A'$  as a function of the driving frequency  $\omega$  shows that  $A'$  reaches a *maximum* at a certain value of  $\omega$ . To find this value, we take the derivative of Eq. (37) with respect to  $\omega$  and set it equal to zero. We find that the maximum value of  $A'$  occurs when

$$\omega = \sqrt{\omega_o^2 - 2\gamma^2}. \quad (39)$$

As in the undamped case, this peaking of the amplitude at a certain angular frequency is called *resonance*, and the graph of  $A'$  as a function of  $\omega$  is called the *resonance curve*. In the light-damping approximation, the peak occurs approximately at  $\omega = \omega_o$ , but in general it occurs at a frequency somewhat less than  $\omega_o$ , as Eq. (39) shows. Also, in the light-damping case the curve is narrow and sharply peaked, while with greater damping it is broader and not as high.

The *phase* of the forced oscillation relative to the driving force is given by the angle  $\phi$  in Eq. (37). When  $\omega$  is close to zero,  $\phi$  is nearly zero and the two are nearly in phase. As  $\omega$  increases,  $\phi$  becomes more and more negative. (I.e., the displacement lags farther and farther behind the force.) When  $\omega = \omega_o$ ,  $\phi = -\pi/2$ , and when  $\omega$  becomes very large,  $\phi$  approaches  $-\pi$  and the force and displacement are nearly a half cycle ( $180^\circ$  or  $\pi$ ) out of phase. This behavior may be compared with that of the *undamped*, driven oscillator, where the phase changes suddenly from 0 to  $-\pi$  when  $\omega = \omega_o$ .

### Light Damping Approximation

When the damping is light (i.e.,  $\gamma \ll \omega_o$ ), the graph of  $A'$  as a function of  $\omega$  is sharply peaked at  $\omega_o$ . In this case we can use an approximate but simpler expression for  $A'$ . We rewrite the first of Eqs. (37):

$$A' = \frac{F_o/m}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2\gamma\omega)^2}} = \frac{F_o/m}{\sqrt{(\omega - \omega_o)^2(\omega + \omega_o)^2 + (2\gamma\omega)^2}}. \quad (40)$$

Then, because the function is appreciably different from zero only near  $\omega = \omega_o$ , we can replace  $\omega$  by  $\omega_o$  in Eq. (40) everywhere except in the term containing  $(\omega - \omega_o)$ . The result is

$$A' \cong \frac{F_o/m}{\sqrt{(\omega - \omega_o)^2(\omega_o + \omega_o)^2 + (2\gamma\omega_o)^2}} = \frac{F_o/2m\omega_o}{\sqrt{(\omega_o - \omega)^2 + \gamma^2}}. \quad (41)$$

In this approximation, the maximum value of  $A'$  occurs exactly at  $\omega = \omega_o$  and is

$$A'_{\max} = \frac{F_o}{2m\gamma\omega_o}. \quad (42)$$

Combining this with Eq. (41), we can also express  $A'$  as

$$A' = \frac{A'_{\max} \gamma}{\sqrt{(\omega - \omega_o)^2 + \gamma^2}}. \quad (43)$$

To describe the sharpness or breadth of the peak, we can find the frequencies at which  $A'$  drops to  $1/\sqrt{2}$  of its peak value as the frequency  $\omega$  of the driving force is varied. Inspection of Eq. (43) shows that this occurs when

$$(\omega - \omega_o)^2 = \gamma^2, \quad \text{or} \quad \omega = \omega_o \pm \gamma. \quad (44)$$

Thus the "width" of the curve (at the points where  $A' = A'_{\max}/\sqrt{2}$ ) is  $2\gamma$ . A more general measure of the sharpness of the peak is the *fractional* width, that is, the ratio of the width ( $2\gamma$ ) to the value of  $\omega$  at the peak (that is,  $\omega_o$ ). We see that the fractional width is simply  $2\gamma/\omega_o$ . This is just the reciprocal of the quantity  $Q$  defined by Eq. (29). Large  $Q$  corresponds to light damping and a sharply peaked resonance curve, and small  $Q$  corresponds to heavy damping and a relatively flat resonance curve. Thus the free-oscillation behavior and the forced-oscillation behavior are closely related.

### Initial Conditions

As with the *undamped*, driven oscillator, Eq. (36) does not represent the *complete* solution of the equation; it is a *particular* solution corresponding to the non-homogeneous term representing the driving force. To obtain the complete solution, we must add to this particular solution the general solution of the corresponding homogeneous equation, which is Eq. (14) (for the underdamped case). Thus the most general solution of Eq. (35), which includes all the possible motions of the system, has the form

$$x = e^{-\gamma t} (A \cos \omega_d t + B \sin \omega_d t) + A' \cos(\omega t + \phi). \quad (45)$$

In this expression, the constants  $A$  and  $B$  are determined by the initial conditions  $x_o$  and  $v_o$ , while  $A'$  depends on the frequency  $\omega$  and amplitude  $F_o$  of the driving force. The first part decays with time and is often called the *transient* part of the solution. The second part is independent of initial conditions, but it persists as long as the driving force is present. It is called the *steady-state* part of the solution.

Equations (15) are not valid for forced oscillations. The complete solution given by Eq. (45) must be used to determine the constants  $A$  and  $B$  in the transient part of the solution in terms of the initial conditions,  $x_o$  and  $v_o$ . The amplitude  $A'$  and phase  $\phi$  of the forced oscillation (or steady-state motion) are determined by the amplitude  $F_o$  of the driving force and its angular frequency  $\omega$ , *not* by the initial conditions. That is,  $A'$  and  $\phi$  *do not* depend on  $x_o$  and  $v_o$ .

The following table summarizes the relationships of the various parts of the general solution, for the undamped and underdamped cases.

Particular Solution of Complete Equation	+	General Solution of Complementary Equation	
$x = \frac{F_o/m}{\omega_o^2 - \omega^2} \cos \omega t$	+	$A \cos \omega_o t + B \sin \omega_o t$	(no damping)
$x = \frac{F_o/m}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2\gamma\omega)^2}} \cos(\omega t + \phi)$	+	$e^{-\gamma t} (A \cos \omega_d t + B \sin \omega_d t)$	(underdamped)
forced or driven oscillation		free oscillation	
depends directly on driving force		independent of driving force	
no arbitrary constants		two arbitrary constants ( $A$ and $B$ )	
independent of initial conditions		depends on initial conditions	
"steady-state" solution (doesn't decay with time)		"transient" solution (dies out with time when damping is present)	

### Complex Number Representation

Many calculations with the harmonic oscillator can be simplified by use of a complex-number representation. An example is the derivation of Eqs. (37). A straightforward, though tedious way to derive these relations is first to expand the expression in Eq. (36) using the cosine sum identity, and then substitute this function and its derivatives into Eq. (35). The sum of cosine terms on the left side must equal the cosine term on the right, and the sum of sine terms on the left side must equal zero. Working out these relations results in two simultaneous equations for  $A'$  and  $\tan \phi$  that may be solved to obtain Eqs. (37).

A simpler method is to use the relation between sine and cosine functions and the exponential function with imaginary argument. (Recall that the imaginary unit  $i$  is defined as  $i = \sqrt{-1}$ .) This relation, called Euler's formula, states that

$$e^{ix} = \cos x + i \sin x. \quad (46)$$

That is,  $e^{ix}$  is a complex quantity with real part  $\cos x$  and imaginary part  $\sin x$ . (We'll study the basis of this relation in greater detail in Section 12.)

Thus Eq. (36) is the real part of the complex function  $A' e^{i(\omega t + \phi)}$ , and the right side of Eq. (35) is the real part of  $(F_o/m) e^{i\omega t}$ . So if we take

$$x = A' e^{i(\omega t + \phi)} \quad (47)$$

(where  $A'$  is a *real* positive constant) and substitute this expression into the equation

$$\ddot{x} + 2\gamma\dot{x} + \omega_o^2 x = \frac{F_o}{m} e^{i\omega t}, \quad (48)$$

then the real part of the function  $x$  is a solution of Eq. (35) if Eq. (47) is a solution of Eq. (48). Substituting Eq. (47) and its derivatives into Eq. (48), we find

$$(-\omega^2 + 2\gamma\omega i + \omega_o^2) A' e^{i(\omega t + \phi)} = \frac{F_o}{m} e^{i\omega t}. \quad (49)$$

We divide out the common factor  $e^{i\omega t}$  and re-arrange the result, to obtain

$$A' e^{i\phi} = \frac{F_o/m}{\omega_o^2 - \omega^2 + 2\gamma\omega i}. \quad (50)$$

Invert this and expand the exponential function using Euler's formula, Eq. (46):

$$\frac{1}{A'} e^{-i\phi} = \frac{1}{A'} (\cos \phi - i \sin \phi) = \frac{m}{F_o} (\omega_o^2 - \omega^2 + 2\gamma\omega i). \quad (51)$$

Equate real and imaginary parts of this equation to obtain

$$\frac{1}{A'} \cos \phi = \frac{m}{F_o} (\omega_o^2 - \omega^2), \quad \frac{1}{A'} \sin \phi = \frac{m}{F_o} (-2\gamma\omega). \quad (52)$$

Squaring each of Eqs. (52), adding, and re-arranging gives

$$\frac{1}{A'^2} (\cos^2 \varphi + \sin^2 \varphi) = \frac{m^2}{F_o^2} \left[ (\omega^2 - \omega_o^2)^2 + (2\gamma\omega)^2 \right], \quad \text{and}$$

$$A' = \frac{F_o/m}{\sqrt{(\omega^2 - \omega_o^2)^2 + (2\gamma\omega)^2}}, \quad (53)$$

which is the first of Eqs. (37). Dividing the second of Eqs. (52) by the first gives

$$\tan \varphi = \frac{2\gamma\omega}{\omega^2 - \omega_o^2}, \quad (54)$$

which is the second of Eqs. (37).

We see that the angle  $\varphi$  represents the *phase* of the forced oscillation relative to the phase of the driving force. The tangent of this angle is the ratio of imaginary to real part of the solution. This relationship recurs many times in analysis of vibrations, a-c circuits, optics, and other areas of physics.

We stress again that the use of complex numbers is not *necessary*; Eqs. (37) can be derived without use of complex quantities. But using complex quantities often simplifies greatly the phase and magnitude relationships among sinusoidally varying quantities.

