33-232 **Exam II** **October 13, 2000**

1. (25 points = 5 + 10 + 10) Name______________________

1	
2	
3	
4	
total	

A body with mass $m = 0.500$ kg moves along a straight line under the action of a force $F = -kx$, where x is the displacement from the equilibrium position and k is a constant. The body is displaced 0.400 m (in the $+x$ direction) from its equilibrium position and given an initial velocity of +1.50 m/s. It oscillates with an angular frequency of 5.00 s^{-1}. There is no driving force.

a) Determine the constant k. Show explicitly that your result has the correct units.

If this is the only force, then there is no damping.

$$\omega_0 = \sqrt{\frac{k}{m}} \Rightarrow k = \omega_0^2 m$$

$$= (5.00\,s^{-1})^2 (0.500\,kg) = \boxed{12.5\ kg/s^2}$$

Also, units of k must be $\frac{N}{m} = \frac{kg\,m/s^2}{m} = kg/s^2$, consistent with above.

b) Obtain an equation for x as a function of t. Determine numerical values for all constants in the equation.

$$x = A\cos\omega_0 t + B\sin\omega_0 t \Rightarrow x_0 = A$$

$$v = -A\omega_0 \sin\omega_0 t + B\omega_0\cos\omega_0 t \Rightarrow v_0 = B\omega_0$$

So $A = x_0 = 0.400\,m$, $B = \frac{v_0}{\omega_0} = \frac{1.50\,m/s}{5.00\,s^{-1}} = 0.300\,m$

$$\boxed{x = (0.400\,m)\cos(5.00\,s^{-1} t) + (0.300\,m)\sin(5.00\,s^{-1} t)}$$

(continued)

1. (continued)

c) Determine the body's maximum distance from the equilibrium position during the motion.

From energy conservation, $\frac{1}{2} m v_0^2 + \frac{1}{2} k x_0^2 = \frac{1}{2} k x_{max}^2$

$$x_{max} = \sqrt{\frac{m}{k} v_0^2 + x_0^2} = \sqrt{\frac{v_0^2}{\omega_0^2} + x_0^2}$$

$$= \sqrt{\frac{(1.50\ m/s)^2}{(5.00\ s^{-1})^2} + (0.400\ m)^2}$$

$$\boxed{x_{max} = 0.500\ m}$$

2. (25 points = 5 + 10 + 10)

For the system in Problem 1, with the same values of m and k, we now add a shock absorber that applies a damping force $F = -bv$, where b is a constant.

a) Determine the value of b that would be needed to make the system critically damped. Show explicitly that your answer has the correct units.

For critical damping, $\gamma = \omega_0$.

also $\gamma = \frac{b}{2m} \Rightarrow b = 2m\gamma = 2m\omega_0$

$$\boxed{b = 2(0.500\ kg)(5.00\ s^{-1}) = 5.00\ kg/s}$$

(Unit analysis same as for Prob. 1(a).)

(continued)

2. (continued)

b) If the constant b has the value found in part (a), and if the initial conditions are the same as for Problem 1, obtain an equation for x as a function of t. Determine numerical values for all constants in your equation.

$x_0 = 0.400\,m$

$v_0 = 1.50\,m/s$

$\gamma = 5.00\,s^{-1}$

From Eq. (16), $x = (A + Bt)e^{-\gamma t}$

From Eq. (17), $A = x_0$, $B = \gamma x_0 + v_0$

$A = 0.400\,m$, $B = (5.00\,s^{-1})(0.400\,m) + 1.50\,m/s$

$= 3.50\,m/s$

$$\boxed{x = \left[0.400\,m + (3.50\,m/s)t\right] e^{-(5.00\,s^{-1})t}}$$

c) If we use a different shock absorber, with $b = 3.00$ (with appropriate units), show that the system is underdamped, and find the period T of damped oscillations.

As $b = 3.00\,kg/s$,

$$\gamma = \frac{b}{2m} = \frac{3.00\,kg/s}{2(0.500\,kg)} = 3.00\,s^{-1}$$

Since $\omega_0 = 5.00\,s^{-1}$, $\gamma < \omega_0$,

and system is underdamped

From Eq. (15), $\omega_d = \sqrt{\omega_0^2 - \gamma^2}$

$\omega_d = \sqrt{(5.00\,s^{-1})^2 - (3.00\,s^{-1})^2}$

$$\boxed{\omega_d = 4.00\,s^{-1}}$$

3. (25 points = 10 + 10 + 5)

A particle with mass $m = 2.00$ kg moves along a straight line under the action of a spring force $F = -kx$, where $k = 5.00$ N/m, and a damping force given by

$$F = -b\left(1 - \frac{x^2}{a^2}\right)v, \qquad \text{where } a = 0.200 \text{ m} \quad \text{and} \quad b = 0.200 \text{ kg/s}.$$

a) Obtain the differential equation for x as a function of time. Express all the variables in terms of x and its derivatives. Do not attempt to solve the equation.

$$m\ddot{x} = -kx - b\left(1 - \frac{x^2}{a^2}\right)\dot{x}$$

$$(2.00\text{kg})\ddot{x} = -(5.00\text{N/m})x - (0.200\text{kg/s})\left(1 - \frac{x^2}{(0.200\text{m})^2}\right)\dot{x}$$

b) Write the Maple code needed to obtain a numerical solution of the differential equation and plot a graph of x as a function of t. Specify numerical values for all constants in the equation.

```
restart;
diffeq:= 2*diff(x(t),t$2) = -5*x(t)
        -0.2*(1-(x(t)/0.2)^2)*diff(x(t),t);
init1:= x(0)=x0;  init2:= D(x)(0) = v0
        # numerical values for x0 and v0
          must be substituted
solution:= dsolve({diffeq,init1,init2}, x(t), numeri
with(plots, odeplot);
odeplot(solution, 0..10);
        # experiment to find an appropriate
          range for t.
```

(continued)

3. (continued)

c) Discuss qualitatively the motion you expect to occur. E.g., does the particle eventually come to rest, or does it keep moving forever, or what?

The motion is oscillating because of the $-kx$ restoring force. When $|x|<a$ the velocity dependent force has the same direction as v and hence adds energy to system. When $|x|>a$ it is more like an ordinary damping force. So we expect an oscillation that doesn't damp out, with amplitude somewhere around a.

4. (25 points = 15 + 10)

a) A damped harmonic oscillator has an angular frequency of free oscillations of $500\ s^{-1}$. It is set into motion, and after 10.0 s the amplitude of vibration has decreased to $1/e$ of its initial value. Determine ω_o, γ, and Q for this system; state any approximations you make.

$\omega_d = 500\ s^{-1}$ $\quad e^{-\gamma(10.0\,s)} = e^{-1} \Rightarrow \gamma = 0.100\ s^{-1}$

From Eq. (15) (or wherever), $\omega_o^2 = \omega_d^2 + \gamma^2$

Here $\gamma \ll \omega_d$ and $\omega_d = \omega_o$ (accurate to 3 significant figs).

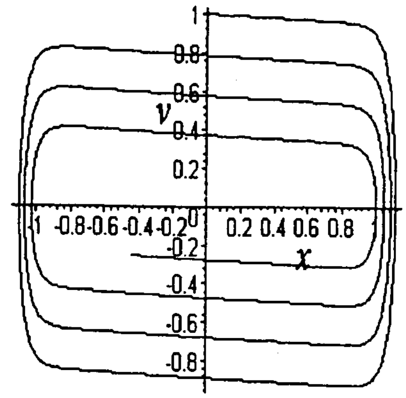
$$Q = \frac{\omega_o}{2\gamma} = \frac{500\ s^{-1}}{2(0.100\ s^{-1})} = 2500$$

$$\boxed{\begin{aligned} \omega_o &= 500\ s^{-1} \\ \gamma &= 0.100\ s^{-1} \\ Q &= 2500 \end{aligned}}$$

(continued)

4. (continued)

b) For the phase plot shown, describe the characteristics of the system as fully as you can. Discuss how the main features of the plot are related to these characteristics. In particular, you may want to comment on the significance of the long straight portions of the curves, their slopes, and the sharp curves at their ends



1. The straight parts are nearly flat ∴ nearly constant v except near $x = \pm 1$. ∴ Small force in middle region

2. as $x \to \pm 1$, force rises very rapidly, causing large acceleration and sudden change in v

3. On straight parts, $|v|$ decreases continuously, showing a velocity-dependent force that is always opposite in direction to v and hence always acts as a damping force.

Useful Equations:

$$\sqrt{\frac{k}{m}}\,T = 2\pi \qquad \text{or} \qquad T = 2\pi\sqrt{\frac{m}{k}}. \tag{6}$$

$$f = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}, \qquad \omega_o = \sqrt{\frac{k}{m}}, \qquad \gamma = \frac{b}{2m}. \tag{7}$$

$$\ddot{x} + 2\gamma\dot{x} + \omega_o{}^2 x = 0. \tag{11}$$

$$x = Ae^{-(\gamma+\gamma_d)t} + Be^{-(\gamma-\gamma_d)t} = e^{-\gamma t}\left(Ae^{-\gamma_d t} + Be^{\gamma_d t}\right), \tag{12}$$

$$A = -\frac{(\gamma - \gamma_d)x_o + v_o}{2\gamma_d}, \qquad B = \frac{(\gamma + \gamma_d)x_o + v_o}{2\gamma_d}. \tag{13}$$

$$x = e^{-\gamma t}\left(A\cos\omega_d t + B\sin\omega_d t\right), \tag{14}$$

$$A = x_o, \qquad B = \frac{v_o + \gamma x_o}{\sqrt{\omega_o{}^2 - \gamma^2}} = \frac{v_o + \gamma x_o}{\omega_d}. \tag{15}$$

$$x = (A + Bt)e^{-\gamma t}. \tag{16}$$

$$A = x_o, \qquad B = \gamma x_o + v_o. \tag{17}$$

$$\left|\frac{\Delta E}{E}\right| \cong \frac{4\pi\gamma}{\omega_d} \cong \frac{4\pi\gamma}{\omega_o}. \tag{27}$$

$$\left|\frac{\Delta E\ (\text{per radian})}{E}\right| = \frac{2\gamma}{\omega_o}. \tag{28}$$

$$Q = \frac{\omega_o}{2\gamma}. \tag{29}$$

$$E = E_o e^{-2\gamma t}. \tag{30}$$

$$A' = \frac{F_o/m}{\omega_o{}^2 - \omega^2}; \qquad \text{then} \qquad x = \frac{F_o/m}{\omega_o{}^2 - \omega^2}\cos\omega t. \tag{33}$$

$$A' = \frac{F_o/m}{\sqrt{\left(\omega^2 - \omega_o{}^2\right)^2 + (2\gamma\omega)^2}}, \qquad \tan\varphi = \frac{2\gamma\omega}{\omega^2 - \omega_o{}^2}. \tag{37}$$

$$x = e^{-\gamma t}\left(A\cos\omega_d t + B\sin\omega_d t\right) + A'\cos(\omega t + \varphi). \tag{45}$$