1. (30 points = 6 + 6 + 6 + 6 + 6) Name

The viscosity of motor oil is sometimes measured by dropping a steel ball with mass \( m \) and radius \( R \) vertically through the oil. The resisting force of the oil is approximately proportional to the speed \( v \) of the ball, and its magnitude is given by \( F = 6\pi\eta Rv \), where \( \eta \) is the viscosity of the oil.

a) Set up the differential equation for \( v \) as a function of time. Be sure to define your coordinate system carefully. Without solving the differential equation, obtain an expression for the terminal velocity \( v_T \) in terms of the other constants.

\[
\begin{align*}
\text{Direction Down, from } \sum F &= m a, \\
m g - 6\pi \eta R v &= m \frac{dv}{dt}
\end{align*}
\]

Terminal velocity is reached when

\[
\frac{dv}{dt} = 0 \Rightarrow m g - 6\pi \eta R v_T = 0
\]

\[
\begin{align*}
v_T &= \frac{mg}{6\pi\eta R} \quad \text{Set } 6\pi\eta R = b, \\
\text{Then } v_T &= \frac{mg}{b}
\end{align*}
\]

b) Discuss whether the way \( v_T \) depends on the constants \( m, g, R, \) and \( \eta \) in your expression seems intuitively reasonable.

Yes: greater mass \( \rightarrow \) greater speed

greater \( g \) \( \rightarrow \) greater weight \( \rightarrow \) greater speed

greater \( R \) \( \rightarrow \) more resisting force \( \rightarrow \) smaller speed

greater \( \eta \) \( \rightarrow \) greater resisting force \( \rightarrow \) smaller speed

(continued)
1. (continued)

c) Determine the SI units of the viscosity $\eta$, using only kilograms, meters, and seconds.

\[
\frac{F}{2\pi R v} = \frac{kg \cdot m/s^2}{m \cdot (m/s)} = \frac{kg}{m \cdot s}
\]

d) If the velocity of the ball is zero at time $t=0$, derive an expression for $v$ as a function of time.

\[\text{Set } b = 2\pi \eta R; \]
\[
mg - bv = \frac{mv}{b} \frac{dv}{dt}, \quad \frac{dv}{dt} = \frac{g}{b} - \frac{v}{b}
\]
\[-\frac{m}{b} \left( g - \frac{v}{b} \right) = vt + c
\]
\[g - \frac{v}{b} = e^{-\frac{v}{bv}} e^{-\frac{v}{b} t} \quad \Longleftrightarrow \quad v = 0 \text{ at } t = 0 \Rightarrow g = e^{-\frac{v}{b} t}
\]
\[v = \frac{mg}{b} \left( 1 - e^{-\frac{v}{b} t} \right)
\]

e) In (d), derive an expression for the time required for the velocity to approach within $1/e$ of its final value.

\[v_T - v = \frac{1}{e} v_T
\]
\[\frac{mg}{b} - \frac{mg}{b} \left( 1 - e^{-\frac{v}{b} t} \right) = e^{-i \left( \frac{mg}{b} \right)}
\]
\[
\text{Simplify: } \quad e^{-\frac{v}{b} t} = e^{-i} \Rightarrow t = \frac{m}{b}
\]
("decay constant")
2. 

In some unusual circumstances a projectile flying through the air may experience an air-resistance force with magnitude approximately proportional to the $3/2$ power of the speed $v$. That is, the magnitude of $F$ is given by $F = b v^{3/2}$, where $b$ is a constant. Assume there is also a constant gravitational force with magnitude $mg$.

a) Define a reasonable coordinate system, and derive expressions for the $x$ and $y$ components of the air-resistance force acting on the projectile, in terms of the time derivatives of $x$ and $y$ and the constants.

$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$F_x = -b v^{3/2} \cos \theta$$

$$= -b v^{3/2} \frac{v_x}{v} = -b v_x v^{1/2}$$

$$F_y = -b \left(\frac{dx}{dt}\right) \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/4}$$

$$F_y = -b \left(\frac{dy}{dt}\right) \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/4}$$

b) Write differential equations for the $x$ and $y$ coordinates, in conventional mathematical notation.

$$-b \left(\frac{dx}{dt}\right) \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/4} = \mu \frac{d^2 x}{dt^2}$$

$$-b \left(\frac{dy}{dt}\right) \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/4} = \mu \frac{d^2 y}{dt^2}$$

(continued)
c) Re-write the equations in (b) in Maple language.

\[ u' = \left( \frac{\partial f(x(t), t)}{\partial x} x + \frac{\partial f(y(t), t)}{\partial y} y \right) \Lambda (1/2) \]

\[ \frac{\partial f}{\partial x} x = -b \cdot \frac{\partial f(x(t), t)}{\partial x} u \Lambda (1/2) = m \cdot \frac{\partial f(x(t), t)}{\partial x} y \]

\[ \frac{\partial f}{\partial y} y = -b \cdot \frac{\partial f(y(t), t)}{\partial y} u \Lambda (1/2) - m \cdot g \]

\[ = m \cdot \frac{\partial f(y(t), t)}{\partial y} y \Lambda (1/2) \]

d) Write whatever additional Maple statements are required in order to solve the differential equations.

\[
\text{init1} := x(0) = 0; \quad \text{init2} := y(0) = 0;
\]

\[
\text{init3} := (x(t))(0) = 0; \quad \text{init4} := (y(t))(0) = 0;
\]

\[
b := 2; \quad m := 50; \quad g := 10;
\]

\[
(\text{arbitrary values for initial conditions and parameters})
\]

\[
\text{sol} := \text{dsolve}\left( \{\text{diff}(x, t), \text{diff}(y, t), \text{init1}, \text{init2}, \text{init3}, \text{init4}, \{x(t), y(t), z(t), \text{numeric} \} \right)
\]
3. (30 points = 6 + 6 + 6 + 6 + 6)

A particle with mass \( m \) moves in a straight line along the positive half of the \( x \) axis. It is acted on by a force that depends on \( x \). The force is described by a potential energy function \( V(x) \) given by

\[
V(x) = V_o \left( \frac{x}{a} + \frac{a}{x} \right), \quad \text{where } V_o \text{ and } a \text{ are positive constants.}
\]

a) Derive an expression for the force \( F(x) \) on the particle, as a function of \( x \).

\[
F = - \frac{dV}{dx} = -V_o \left( \frac{1}{a} - \frac{a}{x^2} \right)
\]

b) Show that there is one equilibrium point, and derive an expression for the value of \( x \) at this point.

At equilibrium, \( F = 0 \)

so \( \frac{1}{a} - \frac{a}{x^2} = 0 \Rightarrow x = a \)

(discard root \( x = -a \) because according to problem statement, \( x > 0 \).)

\[\text{(continued)}\]

c) Show that the point found in (b) is a point of stable equilibrium.

Stable is \( \frac{d^2V}{dx^2} > 0 \) at equilibrium point

\[
\frac{dV}{dx} = V_o \left( \frac{1}{a} - \frac{a}{x^2} \right); \quad \frac{d^2V}{dx^2} = V_o \left( \frac{2a}{x^3} \right).
\]

at \( x = a \), \( \frac{d^2V}{dx^2} = V_o \left( \frac{2a}{a^3} \right) = \frac{2V_o}{a^2} \)

Positive \( \rightarrow \) stable equilibrium.
3. (continued)

d) Derive an expression for the potential energy at the equilibrium point.

\[ x = a, \quad V = V_0 \left( \frac{a}{x} + \frac{a}{\overline{x}} \right) = 2V_0. \]

e) If the total energy of the particle (kinetic plus potential) is equal to \( \frac{5}{2}V_0 \), find the maximum and minimum values of \( x \) in the particle's motion.

At max and min points \( K = 0 \) and total energy = \( V \), so

\[ V_0 \left( \frac{x}{a} + \frac{a}{x} \right) = \frac{5}{2} V_0. \]

Solve for \( x \): first multiply by \( \frac{a}{V_0} \):

\[ x^2 + a^2 = \frac{5}{2} a x, \]

\[ (x - 2a)(x - \frac{a}{2}) = 0 \]

Roots: \( x = \frac{a}{2} \) (min)
\( x = 2a \) (max)