33-232 **Exam I** **September 22, 2000**

1. (30 points = 6 + 6 + 6 + 6 + 6) Name_____

The viscosity of motor oil is sometimes measured by dropping a steel ball with mass m and radius R vertically through the oil. The resisting force of the oil is approximately proportional to the speed v of the ball, and its magnitude is given by $F = 6\pi\eta R v$, where η is the viscosity of the oil.

1	
2	
3	
total	

a) Set up the differential equation for v as a function of time. Be sure to define your coordinate system carefully. Without solving the differential equation, obtain an expression for the terminal velocity v_T in terms of the other constants.

+ direction down. From $\Sigma F = ma$,

$$mg - 6\pi\eta R v = m\frac{dv}{dt}.$$

Terminal velocity is reached when

$$\frac{dv}{dt} = 0 \Rightarrow mg - 6\pi\eta R v_T = 0$$

$$\boxed{v_T = \frac{mg}{6\pi\eta R}}$$

Let $6\pi\eta R = b$, then $v_T = \frac{mg}{b}$

b) Discuss whether the way V_T depends on the constants m, g, R, and η in your expression seems intuitively reasonable.

Yes: greater mass → greater speed
greater g → greater weight → greater speed

greater R → more resisting force → smaller speed

greater η → greater resisting force → smaller speed

(continued)

1. (continued)

c) Determine the SI units of the viscosity η, using only kilograms, meters, and seconds.

$$\eta = \frac{F}{6\pi R v} \qquad F : kg\,m/s^2$$

units : $\dfrac{kg\,m/s^2}{m\,(m/s)} = \boxed{\dfrac{kg}{m \cdot s}}$

d) If the velocity of the ball is zero at time $t = 0$, derive an expression for v as a function of time.

Set $b = 6\pi \eta R$;

$$mg - bv = \frac{m\,dv}{dt}, \quad \frac{dv}{g - \frac{b}{m}v} = dt$$

$$-\frac{m}{b}\ln\left(g - \frac{b}{m}v\right) = t + c$$

$$g - \frac{b}{m}v = e^{-\frac{b}{m}t}e^{-\frac{b}{m}c}; \quad v = 0 \text{ at } t = 0 \Rightarrow g = e^{-\frac{b}{m}t}$$

$$\boxed{v = \frac{mg}{b}\left(1 - e^{-\frac{b}{m}t}\right)}$$

e) In (d), derive an expression for the time required for the velocity to approach within $1/e$ of its final value.

$$v_T - v = \frac{1}{e}v_T$$

$$\frac{mg}{b} - \frac{mg}{b}\left(1 - e^{-\frac{b}{m}t}\right) = e^{-1}\left(\frac{mg}{b}\right)$$

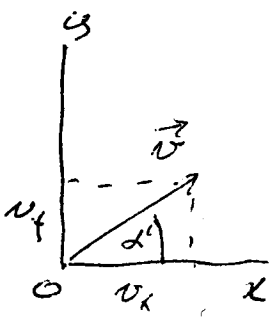
Simplify : $e^{-\frac{b}{m}t} = e^{-1} \Rightarrow \boxed{t = \frac{m}{b}}$

("decay constant")

2. (40 points = 10 + 10 + 10 + 10)

In some unusual circumstances a projectile flying through the air may experience an air-resistance force with magnitude approximately proportional to the 3/2 power of the speed v. That is, the magnitude of F is given by $F = bv^{3/2}$, where b is a constant. Assume there is also a constant gravitational force with magnitude mg.

a) Define a reasonable coordinate system, and derive expressions for the x and y components of the air-resistance force acting on the projectile, in terms of the time derivatives of x and y and the constants.



$$v = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$F_x = -b v^{3/2} \cos\alpha$$

$$= -b v^{3/2} \cdot \frac{v_x}{v} = -b v_x v^{1/2}$$

$$F_x = -b\left(\frac{dx}{dt}\right)\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/4}$$

$$F_y = -b\left(\frac{dy}{dt}\right)\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/4}$$

b) Write differential equations for the x and y coordinates, in conventional mathematical notation.

$$-b\left(\frac{dx}{dt}\right)\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/4} = m\frac{d^2x}{dt^2}$$

$$-b\left(\frac{dy}{dt}\right)\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]^{1/4} - mg = m\frac{d^2y}{dt^2}$$

(continued)

2. (continued)

c) Re-write the equations in (b) in Maple language.

```
v := (diff(x(t),t)^2+diff(y(t),t)^2)^(1/2)

diffeqx := -b*diff(x(t),t)*v^(1/2) = m*diff(x(t,t#2));

diffeqy := -b*diff(y(t),t)*v^(1/2) - m*g
           = m*diff(y(t),t#2);
```

d) Write whatever additional Maple statements are required in order to solve the differential equations.

```
init1:=x(0)=0;    init2:=y(0)=0;

init3:=D(x)(0):=10;   init4:=(y)(0)=20;

b:=2;  m:=80;  g:=10;
```

(arbitrary values for initial conditions and parameters)

```
sol:=dsolve({diffeqx, diffeqy, init1, init2,
  init3, init4}, {x(t), y(t)}, numeric);
```

3. (30 points = 6 + 6 + 6 + 6 + 6)

A particle with mass m moves in a straight line along the positive half of the x axis. It is acted on by a force that depends on x. The force is described by a potential energy function $V(x)$ given by

$$V(x) = V_o\left(\frac{x}{a} + \frac{a}{x}\right), \qquad \text{where } V_o \text{ and } a \text{ are positive constants.}$$

a) Derive an expression for the force $F(x)$ on the particle, as a function of x.

$$F = -\frac{dV}{dx} = -V_o\left(\frac{1}{a} - \frac{a}{x^2}\right)$$

b) Show that there is one equilibrium point, and derive an expression for the value of x at this point.

at equilibrium, $F = 0$

so $\frac{1}{a} - \frac{a}{x^2} = 0 \Rightarrow \boxed{x = a}$

(discard root $x = -a$ because according to problem statement, $x > 0$.)

c) Show that the point found in (b) is a point of *stable* equilibrium.

Stable if $\frac{d^2V}{dx^2} > 0$ at equilibrium point

$\frac{dV}{dx} = V_o\left(\frac{1}{a} - \frac{a}{x^2}\right)$; $\frac{d^2V}{dx^2} = V_o\left(\frac{2a}{x^3}\right)$.

at $x = a$, $\frac{d^2V}{dx^2} = V_o\left(\frac{2a}{a^3}\right) = \frac{2V_o}{a^2}$ (continued)

positive $\rightarrow$ stable equilibrium

3. (continued)

d) Derive an expression for the potential energy at the equilibrium point.

at $x = a$, $V = V_0\left(\frac{a}{a} + \frac{a}{a}\right) = \boxed{2 \cdot V_0}$

e) If the total energy of the particle (kinetic plus potential) is equal to $\frac{5}{2}V_0$, find the maximum and minimum values of x in the particle's motion.

at max and min points $K = 0$ and total energy $= V$, so

$V_0\left(\frac{x}{a} + \frac{a}{x}\right) = \frac{5}{2}V_0$. Solve for x:

first multiply by $\frac{ax}{V_0}$:

$$x^2 + a^2 = \frac{5}{2}ax.$$

$(x - 2a)(x - \frac{a}{2}) = 0$ Roots: $x = \frac{a}{2}$ (min)

$x = 2a$ (max)