**33-231** **Physical Analysis** **Fall 2003**

**Problems:** Set 11 (due Wednesday, November 12, 2003)

**Note**: In Problems 43 through 45, all matrix calculations should be done two ways. First do them by hand, then check them using Maple. In each problem, some intermediate results are given as guideposts, to help you stay on track.

43. This problem concerns the same mechanical system as Problem 38. Use the same coordinates as in Problem 38.

a) Obtain the **M** and **K** matrices, as discussed in class, and write the equations of motion in matrix form, i.e., in the form $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}$.

$$\mathbf{M} = \begin{pmatrix} 3m & 0 \\ 0 & 2m \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}$$

b) Form the matrix $\mathbf{K} - \lambda\mathbf{M}$, obtain the secular determinant $|\mathbf{K} - \lambda\mathbf{M}|$, set it equal to zero, and solve the resulting equation to find the possible values of $\lambda$, i.e., the eigenvalues of the problem. You may want to number them consistently with your work in Problem 38. (I recommend putting them in ascending order, with the smallest one first.) Compare your values of $\lambda$ with your results from Problem 38.

$$\lambda_1 = \omega_1^{\,2} = \frac{k}{6m}, \qquad \lambda_2 = \omega_2^{\,2} = \frac{k}{m}.$$

c) For each eigenvalue find the corresponding eigenvector. Let **a** be the eigenvector corresponding to $\lambda_1$, and let **b** be the eigenvector corresponding to $\lambda_2$. Thus show that **a** and **b** satisfy the equations

$$(\mathbf{K} - \lambda_1\mathbf{M})\mathbf{a} = 0 \qquad \text{and} \qquad (\mathbf{K} - \lambda_2\mathbf{M})\mathbf{b} = 0.$$

$$\mathbf{a} = \begin{pmatrix} 1 \\ \frac{3}{2} \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

d) Form a matrix **A** with the eigenvectors **a** and **b** as columns and express the normal-coordinate transformation as a matrix equation, in the form $\mathbf{x} = \mathbf{A}\mathbf{q}$. Obtain expressions for $x_1$ and $x_2$ in terms of $q_1$ and $q_2$, and compare with your results from Problem 38. Also obtain the inverse transformation $\mathbf{q} = \mathbf{A}^{-1}\mathbf{x}$.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ \frac{3}{2} & -1 \end{pmatrix}, \qquad \begin{aligned} x_1 &= q_1 + q_2, \\ x_2 &= \frac{3}{2}q_1 - q_2. \end{aligned}$$

(continued)

43. (continued)

e) Express the equations of motion (in matrix form) in terms of the normal coordinates, and show that the differential equations for the $q$'s can be separated into an equation containing only $q_1$ and its derivatives, and an equation containing only $q_2$ and its derivatives.

$$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}, \qquad \mathbf{M}\mathbf{A}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{A}\mathbf{q} ,$$

$$3m\ddot{q}_1 + 3m\ddot{q}_2 = -\frac{k}{2}q_1 - 3kq_2 ,$$

$$3m\ddot{q}_1 - 2m\ddot{q}_2 = -\frac{k}{2}q_1 + 2kq_2 .$$

(Add and subtract, with appropriate multipliers, to obtain uncoupled equations.)

f) Determine the normal-mode frequencies from the separated equations for $q_1$ and $q_2$, and compare your results with those from part (b).

44. Repeat the calculations of Problem 43, but using a different coordinate system. Let $x_1$ be the coordinate of mass $3m$ in an inertial frame of reference, as before, and let $x_2$ be the displacement of mass $2m$ relative to mass $3m$, i.e., the elongation of the right-hand spring. Following are guideposts corresponding to those in Problem 43.

a) $$\mathbf{M} = \begin{pmatrix} 3m & 0 \\ 2m & 2m \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} k & -k \\ 0 & k \end{pmatrix}.$$

b) $$\lambda_1 = \frac{k}{6m}, \qquad \lambda_2 = \frac{k}{m}.$$

c) $$\mathbf{a} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

d) $$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \qquad x_1 = 2q_1 + q_2, \qquad x_2 = q_1 - 2q_2$$

e) $$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}, \qquad \mathbf{M}\mathbf{A}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{A}\mathbf{q} ,$$

$$6m\ddot{q}_1 + 3m\ddot{q}_2 = -kq_1 - 3kq_2 ,$$

$$6m\ddot{q}_1 - 2m\ddot{q}_2 = -kq_1 + 2kq_2 .$$

45. Carry out the same calculations as for Problems 43 and 44, for the system of Problem 39. Following are guideposts as before.

a) $$\mathbf{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} = m\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix} = k\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

b)

$$|\mathbf{K} - \lambda\mathbf{M}| = \begin{vmatrix} k - \lambda m & -k & 0 \\ -k & 2k - \lambda m & -k \\ 0 & -k & k - \lambda m \end{vmatrix} = (k - \lambda m)^2(2k - \lambda m) - 2k^2(k - \lambda m).$$

$$\lambda_1 = 0, \quad \lambda_2 = \frac{k}{m}, \quad \lambda_3 = \frac{3k}{m}.$$

c) For $\lambda_1$: $$(\mathbf{K} - \lambda\mathbf{M})\mathbf{a} = \begin{pmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = 0, \qquad \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

For $\lambda_2$, $$(\mathbf{K} - \lambda\mathbf{M})\mathbf{b} = \begin{pmatrix} 0 & -k & 0 \\ -k & k & -k \\ 0 & -k & 0 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = 0, \qquad \mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For $\lambda_3$, $$(\mathbf{K} - \lambda\mathbf{M})\mathbf{c} = \begin{pmatrix} -2k & -k & 0 \\ -k & -k & -k \\ 0 & -k & -2k \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0, \qquad \mathbf{c} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

d) $$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{pmatrix}, \qquad \begin{aligned} x_1 &= q_1 + q_2 + q_3, \\ x_2 &= q_1 - 2q_3, \\ x_3 &= q_1 - q_2 + q_3. \end{aligned}$$

e) $$\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{K}\mathbf{x}, \qquad \mathbf{M}\mathbf{A}\ddot{\mathbf{q}} = -\mathbf{K}\mathbf{A}\mathbf{q},$$

Equations for $q_1$, $q_2$, and $q_3$ are the same as for Problem 39(c). By adding and subtracting multiples of these equations, separated equations for the $q$'s can be obtained, just as in Problem 39(c).